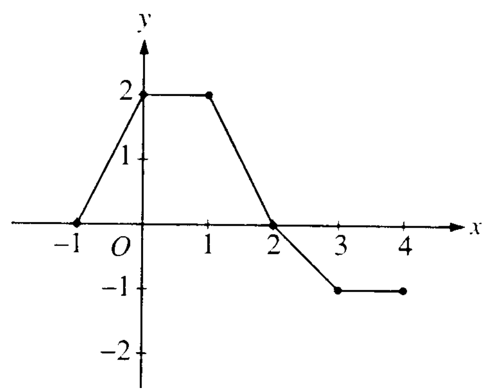
55 Minutes—No Calculator

Note: Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

1. What is the x-coordinate of the point of inflection on the graph of $y = \frac{1}{3}x^3 + 5x^2 + 24$?

(A) 5 (B) 0 (C) $-\frac{10}{3}$ (D) -5 (E) -10



2. The graph of a piecewise-linear function f, for $-1 \le x \le 4$, is shown above. What is the value of $\int_{-1}^{4} f(x)\,dx$?

(A) 1 (B) 2.5 (C) 4 (D) 5.5 (E) 8

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3. $\int_{1}^{2} \frac{1}{x^2}\,dx =$

(A) $-\frac{1}{2}$ (B) $\frac{7}{24}$ (C) $\frac{1}{2}$ (D) 1 (E) $2\ln 2$

4. If f is continuous for $a \le x \le b$ and differentiable for $a < x < b$, which of the following could be false?

(A) $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ for some c such that $a < c < b$.

(B) $f'(c) = 0$ for some c such that $a < c < b$.

(C) f has a minimum value on $a \le x \le b$.

(D) f has a maximum value on $a \le x \le b$.

(E) $\int_a^b f(x)\,dx$ exists.

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5. $\int_0^x \sin t\,dt =$

(A) $\sin x$ (B) $-\cos x$ (C) $\cos x$ (D) $\cos x - 1$ (E) $1 - \cos x$

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6. If $x^2 + xy = 10$, then when $x = 2$, $\dfrac{dy}{dx} =$

(A) $-\dfrac{7}{2}$ (B) -2 (C) $\dfrac{2}{7}$ (D) $\dfrac{3}{2}$ (E) $\dfrac{7}{2}$

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7. $\int_1^e \left(\dfrac{x^2 - 1}{x}\right) dx =$

(A) $e - \dfrac{1}{e}$ (B) $e^2 - e$ (C) $\dfrac{e^2}{2} - e + \dfrac{1}{2}$ (D) $e^2 - 2$ (E) $\dfrac{e^2}{2} - \dfrac{3}{2}$

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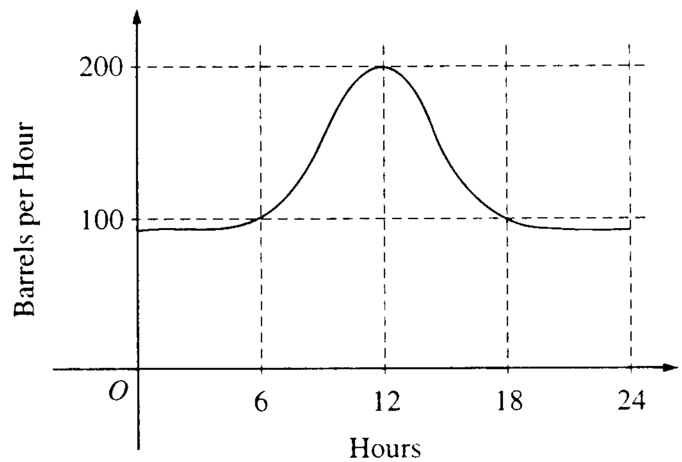
8. Let f and g be differentiable functions with the following properties:

(i) $g(x) > 0$ for all x
(ii) $f(0) = 1$

If $h(x) = f(x)g(x)$ and $h'(x) = f(x)g'(x)$, then $f(x) =$

(A) $f'(x)$ (B) $g(x)$ (C) e^x (D) 0 (E) 1



9. The flow of oil, in barrels per hour, through a pipeline on July 9 is given by the graph shown above. Of the following, which best approximates the total number of barrels of oil that passed through the pipeline that day?

(A) 500 (B) 600 (C) 2,400 (D) 3,000 (E) 4,800

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10. What is the instantaneous rate of change at $x = 2$ of the function f given by $f(x) = \dfrac{x^2 - 2}{x - 1}$?

(A) -2 (B) $\dfrac{1}{6}$ (C) $\dfrac{1}{2}$ (D) 2 (E) 6

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11. If f is a linear function and $0 < a < b$, then $\int_a^b f''(x)\,dx =$

(A) 0 (B) 1 (C) $\dfrac{ab}{2}$ (D) $b - a$ (E) $\dfrac{b^2 - a^2}{2}$

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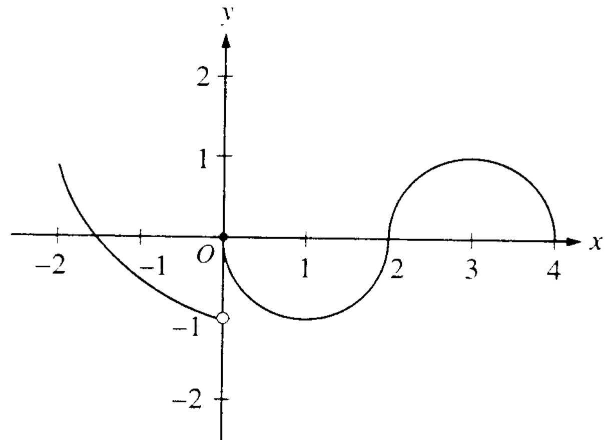
12. If $f(x) = \begin{cases} \ln x & \text{for } 0 < x \le 2 \\ x^2 \ln 2 & \text{for } 2 < x \le 4, \end{cases}$ then $\lim\limits_{x \to 2} f(x)$ is

(A) $\ln 2$ (B) $\ln 8$ (C) $\ln 16$ (D) 4 (E) nonexistent



13. The graph of the function f shown in the figure above has a vertical tangent at the point $(2,0)$ and horizontal tangents at the points $(1,-1)$ and $(3,1)$. For what values of x, $-2 < x < 4$, is f not differentiable?

(A) 0 only (B) 0 and 2 only (C) 1 and 3 only (D) 0, 1, and 3 only (E) 0, 1, 2, and 3

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14. A particle moves along the x-axis so that its position at time t is given by $x(t) = t^2 - 6t + 5$. For what value of t is the velocity of the particle zero?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

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15. If $F(x) = \int_0^x \sqrt{t^3 + 1}\, dt$, then $F'(2) =$

(A) -3 (B) -2 (C) 2 (D) 3 (E) 18

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16. If $f(x) = \sin\left(e^{-x}\right)$, then $f'(x) =$

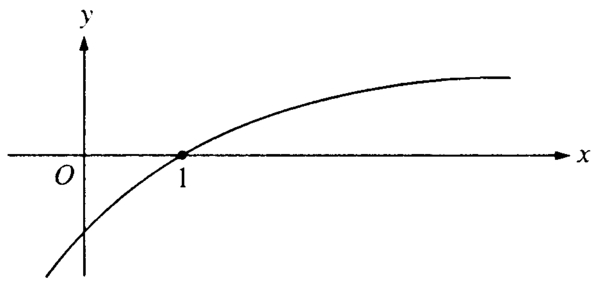
(A) $-\cos(e^{-x})$

(B) $\cos(e^{-x}) + e^{-x}$

(C) $\cos(e^{-x}) - e^{-x}$

(D) $e^{-x}\cos(e^{-x})$

(E) $-e^{-x}\cos(e^{-x})$



17. The graph of a twice-differentiable function f is shown in the figure above. Which of the following is true?

(A) $f(1) < f'(1) < f''(1)$

(B) $f(1) < f''(1) < f'(1)$

(C) $f'(1) < f(1) < f''(1)$

(D) $f''(1) < f(1) < f'(1)$

(E) $f''(1) < f'(1) < f(1)$

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18. An equation of the line tangent to the graph of $y = x + \cos x$ at the point $(0,1)$ is

(A) $y = 2x + 1$ (B) $y = x + 1$ (C) $y = x$ (D) $y = x - 1$ (E) $y = 0$

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19. If $f''(x) = x(x+1)(x-2)^2$, then the graph of f has inflection points when $x =$

(A) –1 only (B) 2 only (C) –1 and 0 only (D) –1 and 2 only (E) –1, 0, and 2 only

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20. What are all values of k for which $\int_{-3}^{k} x^2 dx = 0$?

(A) –3 (B) 0 (C) 3 (D) –3 and 3 (E) –3, 0, and 3

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21. If $\frac{dy}{dt} = ky$ and k is a nonzero constant, then y could be

(A) $2e^{kty}$ (B) $2e^{kt}$ (C) $e^{kt} + 3$ (D) $kty + 5$ (E) $\frac{1}{2}ky^2 + \frac{1}{2}$

22. The function f is given by $f(x) = x^4 + x^2 - 2$. On which of the following intervals is f increasing?

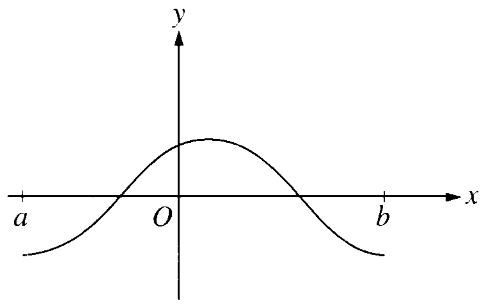
(A) $\left(-\frac{1}{\sqrt{2}}, \infty\right)$

(B) $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

(C) $(0, \infty)$

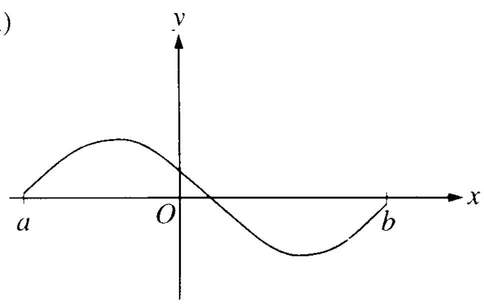
(D) $(-\infty, 0)$

(E) $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$

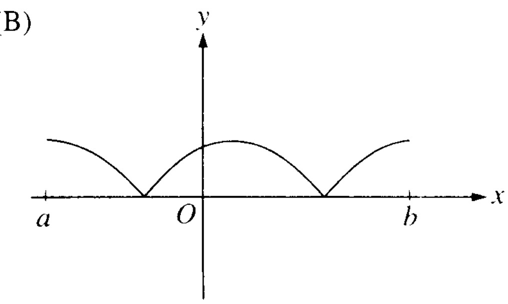


23. The graph of f is shown in the figure above. Which of the following could be the graph of the derivative of f?

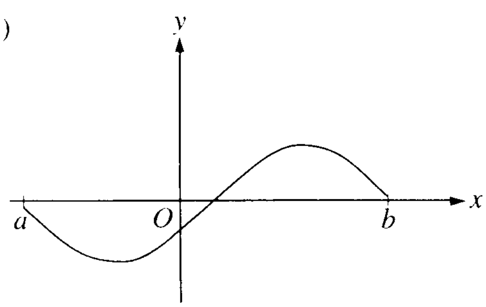
(A)



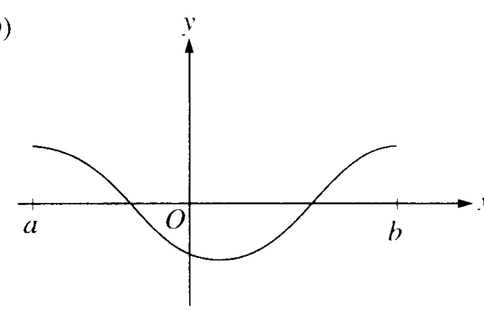
(B)



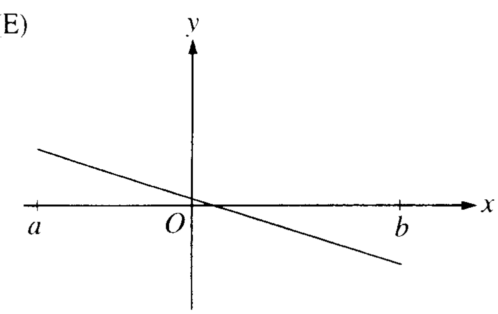
(C)



(D)



(E)



24. The maximum acceleration attained on the interval $0 \le t \le 3$ by the particle whose velocity is given by $v(t) = t^3 - 3t^2 + 12t + 4$ is

(A) 9 (B) 12 (C) 14 (D) 21 (E) 40

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25. What is the area of the region between the graphs of $y = x^2$ and $y = -x$ from $x = 0$ to $x = 2$?

(A) $\frac{2}{3}$ (B) $\frac{8}{3}$ (C) 4 (D) $\frac{14}{3}$ (E) $\frac{16}{3}$

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x	0	1	2
$f(x)$	1	k	2

26. The function f is continuous on the closed interval $[0,2]$ and has values that are given in the table above. The equation $f(x) = \frac{1}{2}$ must have at least two solutions in the interval $[0,2]$ if $k =$

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) 2 (E) 3

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27. What is the average value of $y = x^2\sqrt{x^3+1}$ on the interval $[0,2]$?

(A) $\frac{26}{9}$ (B) $\frac{52}{9}$ (C) $\frac{26}{3}$ (D) $\frac{52}{3}$ (E) 24

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28. If $f(x) = \tan(2x)$, then $f'\left(\frac{\pi}{6}\right) =$

(A) $\sqrt{3}$ (B) $2\sqrt{3}$ (C) 4 (D) $4\sqrt{3}$ (E) 8

1. D $y' = x^2 + 10x$; $y'' = 2x + 10$; y'' changes sign at $x = -5$

2. B $\int_{-1}^{4} f(x)dx = \int_{-1}^{2} f(x)dx + \int_{2}^{4} f(x)dx$

$= \text{Area of trapezoid(1)} - \text{Area of trapezoid(2)} = 4 - 1.5 = 2.5$

3. C $\int_{1}^{2} \frac{1}{x^2} dx = \int_{1}^{2} x^{-2} dx = -x^{-1}\Big|_{1}^{2} = \frac{1}{2}$

4. B This would be false if f was a linear function with non-zero slope.

5. E $\int_{0}^{x} \sin t\, dt = -\cos t\Big|_{0}^{x} = -\cos x - (-\cos 0) = -\cos x + 1 = 1 - \cos x$

6. A Substitute $x = 2$ into the equation to find $y = 3$. Taking the derivative implicitly gives $\frac{d}{dx}\left(x^2 + xy\right) = 2x + xy' + y = 0$. Substitute for x and y and solve for y'.

$4 + 2y' + 3 = 0;\ y' = -\frac{7}{2}$

7. E $\int_{1}^{e} \frac{x^2-1}{x}\, dx = \int_{1}^{e} x - \frac{1}{x}\, dx = \left(\frac{1}{2}x^2 - \ln x\right)\Big|_{1}^{e} = \left(\frac{1}{2}e^2 - 1\right) - \left(\frac{1}{2} - 0\right) = \frac{1}{2}e^2 - \frac{3}{2}$

8. E $h(x) = f(x)g(x)$ so, $h'(x) = f'(x)g(x) + f(x)g'(x)$. It is given that $h'(x) = f(x)g'(x)$. Thus, $f'(x)g(x) = 0$. Since $g(x) > 0$ for all x, $f'(x) = 0$. This means that f is constant. It is given that $f(0) = 1$, therefore $f(x) = 1$.

9. D Let $r(t)$ be the rate of oil flow as given by the graph, where t is measured in hours. The total number of barrels is given by $\int_{0}^{24} r(t)dt$. This can be approximated by counting the squares below the curve and above the horizontal axis. There are approximately five squares with area 600 barrels. Thus the total is about $3,000$ barrels.

10. D $f'(x) = \frac{(x-1)(2x) - (x^2-2)(1)}{(x-1)^2}$; $f'(2) = \frac{(2-1)(4) - (4-2)(1)}{(2-1)^2} = 2$

11. A Since f is linear, its second derivative is zero. The integral gives the area of a rectangle with zero height and width $(b - a)$. This area is zero.

12. E $\lim_{x\to 2^-} f(x) = \ln 2 \neq 4\ln 2 = \lim_{x\to 2^+} f(x)$. Therefore the limit does not exist.

13. B At $x = 0$ and $x = 2$ only. The graph has a non-vertical tangent line at every other point in the interval and so has a derivative at each of these other x's.

14. C $v(t) = 2t - 6$; $v(t) = 0$ for $t = 3$

15 D By the Fundamental Theorem of Calculus, $F'(x) = \sqrt{x^3+1}$, thus $F'(2) = \sqrt{2^3+1} = \sqrt{9} = 3$.

16. E $f'(x) = \cos(e^{-x}) \cdot \frac{d}{dx}(e^{-x}) = \cos(e^{-x})\left(e^{-x} \cdot \frac{d}{dx}(-x)\right) = -e^{-x}\cos(e^{-x})$

17. D From the graph $f(1) = 0$. Since $f'(1)$ represents the slope of the graph at $x = 1$, $f'(1) > 0$. Also, since $f''(1)$ represents the concavity of the graph at $x = 1$, $f''(1) < 0$.

18. B $y' = 1 - \sin x$ so $y'(0) = 1$ and the line with slope 1 containing the point (0,1) is $y = x + 1$.

19. C Points of inflection occur where f'' changes sign. This is only at $x = 0$ and $x = -1$. There is no sign change at $x = 2$.

20. A $\int_{-3}^{k} x^2\,dx = \frac{1}{3}x^3\Big|_{-3}^{k} = \frac{1}{3}\left(k^3 - (-3)^3\right) = \frac{1}{3}\left(k^3 + 27\right) = 0$ only when $k = -3$.

21. B The solution to this differential equation is known to be of the form $y = y(0)\cdot e^{kt}$. Option (B) is the only one of this form. If you do not remember the form of the solution, then separate the variables and antidifferentiate.
$\frac{dy}{y} = k\,dt$; $\ln|y| = kt + c_1$; $|y| = e^{kt+c_1} = e^{kt}e^{c_1}$; $y = ce^{kt}$.

22. C f is increasing on any interval where $f'(x) > 0$. $f'(x) = 4x^3 + 2x = 2x(2x^2+1) > 0$.
Since $(x^2+1) > 0$ for all x, $f'(x) > 0$ whenever $x > 0$.

23. A The graph shows that f is increasing on an interval (a,c) and decreasing on the interval (c,b), where $a < c < b$. This means the graph of the derivative of f is positive on the interval (a,c) and negative on the interval (c,b), so the answer is (A) or (E). The derivative is not (E), however, since then the graph of f would be concave down for the entire interval.

24. D The maximum acceleration will occur when its derivative changes from positive to negative or at an endpoint of the interval. $a(t) = v'(t) = 3t^2 - 6t + 12 = 3(t^2 - 2t + 4)$ which is always positive. Thus the acceleration is always increasing. The maximum must occur at $t = 3$ where $a(3) = 21$

25. D The area is given by $\int_0^2 x^2 - (-x)\,dx = \left(\frac{1}{3}x^3 + \frac{1}{2}x^2\right)\Big|_0^2 = \frac{8}{3} + 2 = \frac{14}{3}$.

26. A Any value of k less than 1/2 will require the function to assume the value of 1/2 at least twice because of the Intermediate Value Theorem on the intervals [0,1] and [1,2]. Hence $k = 0$ is the only option.

27. A $\frac{1}{2}\int_0^2 x^2\sqrt{x^3+1}\,dx = \frac{1}{2}\int_0^2 (x^3+1)^{\frac{1}{2}}\left(\frac{1}{3}\cdot 3x^2\right)dx = \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{2}{3}(x^3+1)^{\frac{3}{2}}\Big|_0^2 = \frac{1}{9}\left(9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) = \frac{26}{9}$

28. E $f'(x) = \sec^2(2x)\cdot\frac{d}{dx}(2x) = 2\sec^2(2x)$; $f'\left(\frac{\pi}{6}\right) = 2\sec^2\left(\frac{\pi}{3}\right) = 2(4) = 8$