90 Minutes—No Calculator

Notes: (1) In this examination, ln x denotes the natural logarithm of x (that is, logarithm to the base e).

(2) Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

1. $\int_1^2 x^{-3}\,dx =$

(A) $-\frac{7}{8}$ (B) $-\frac{3}{4}$ (C) $\frac{15}{64}$ (D) $\frac{3}{8}$ (E) $\frac{15}{16}$

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2. If $f(x) = (2x+1)^4$, then the 4th derivative of $f(x)$ at $x = 0$ is

(A) 0 (B) 24 (C) 48 (D) 240 (E) 384

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3. If $y = \frac{3}{4+x^2}$, then $\frac{dy}{dx} =$

(A) $\frac{-6x}{\left(4+x^2\right)^2}$ (B) $\frac{3x}{\left(4+x^2\right)^2}$ (C) $\frac{6x}{\left(4+x^2\right)^2}$ (D) $\frac{-3}{\left(4+x^2\right)^2}$ (E) $\frac{3}{2x}$

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4. If $\frac{dy}{dx} = \cos(2x)$, then $y =$

(A) $-\frac{1}{2}\cos(2x)+C$ (B) $-\frac{1}{2}\cos^2(2x)+C$ (C) $\frac{1}{2}\sin(2x)+C$

(D) $\frac{1}{2}\sin^2(2x)+C$ (E) $-\frac{1}{2}\sin(2x)+C$

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5. $\lim_{n\to\infty} \frac{4n^2}{n^2+10{,}000n}$ is

(A) 0 (B) $\frac{1}{2{,}500}$ (C) 1 (D) 4 (E) nonexistent

6. If $f(x) = x$, then $f'(5) =$

(A) 0 (B) $\frac{1}{5}$ (C) 1 (D) 5 (E) $\frac{25}{2}$

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7. Which of the following is equal to $\ln 4$?

(A) $\ln 3 + \ln 1$ (B) $\frac{\ln 8}{\ln 2}$ (C) $\int_1^4 e^t \, dt$ (D) $\int_1^4 \ln x \, dx$ (E) $\int_1^4 \frac{1}{t} \, dt$

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8. The slope of the line tangent to the graph of $y = \ln\left(\frac{x}{2}\right)$ at $x = 4$ is

(A) $\frac{1}{8}$ (B) $\frac{1}{4}$ (C) $\frac{1}{2}$ (D) 1 (E) 4

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9. If $\int_{-1}^{1} e^{-x^2} \, dx = k$, then $\int_{-1}^{0} e^{-x^2} \, dx =$

(A) $-2k$ (B) $-k$ (C) $-\frac{k}{2}$ (D) $\frac{k}{2}$ (E) $2k$

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10. If $y = 10^{\left(x^2-1\right)}$, then $\frac{dy}{dx} =$

(A) $(\ln 10)10^{\left(x^2-1\right)}$ (B) $(2x)10^{\left(x^2-1\right)}$ (C) $\left(x^2-1\right)10^{\left(x^2-2\right)}$

(D) $2x(\ln 10)10^{\left(x^2-1\right)}$ (E) $x^2(\ln 10)10^{\left(x^2-1\right)}$

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11. The position of a particle moving along a straight line at any time t is given by $s(t) = t^2 + 4t + 4$. What is the acceleration of the particle when $t = 4$?

(A) 0 (B) 2 (C) 4 (D) 8 (E) 12

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12. If $f(g(x)) = \ln\left(x^2 + 4\right)$, $f(x) = \ln\left(x^2\right)$, and $g(x) > 0$ for all real x, then $g(x) =$

(A) $\frac{1}{\sqrt{x^2+4}}$ (B) $\frac{1}{x^2+4}$ (C) $\sqrt{x^2+4}$ (D) x^2+4 (E) $x+2$

13. If $x^2+xy+y^3=0$, then, in terms of x and y, $\frac{dy}{dx}=$

(A) $-\frac{2x+y}{x+3y^2}$ (B) $-\frac{x+3y^2}{2x+y}$ (C) $\frac{-2x}{1+3y^2}$ (D) $\frac{-2x}{x+3y^2}$ (E) $-\frac{2x+y}{x+3y^2-1}$

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14. The velocity of a particle moving on a line at time t is $v=3t^{\frac{1}{2}}+5t^{\frac{3}{2}}$ meters per second. How many meters did the particle travel from $t=0$ to $t=4$?

(A) 32 (B) 40 (C) 64 (D) 80 (E) 184

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15. The domain of the function defined by $f(x)=\ln\left(x^2-4\right)$ is the set of all real numbers x such that

(A) $|x|<2$ (B) $|x|\le 2$ (C) $|x|>2$ (D) $|x|\ge 2$ (E) x is a real number

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16. The function defined by $f(x)=x^3-3x^2$ for all real numbers x has a relative maximum at $x=$

(A) -2 (B) 0 (C) 1 (D) 2 (E) 4

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17. $\int_0^1 xe^{-x}dx=$

(A) $1-2e$ (B) -1 (C) $1-2e^{-1}$ (D) 1 (E) $2e-1$

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18. If $y=\cos^2 x-\sin^2 x$, then $y'=$

(A) -1 (B) 0 (C) $-2\sin(2x)$ (D) $-2(\cos x+\sin x)$ (E) $2(\cos x-\sin x)$

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19. If $f(x_1)+f(x_2)=f(x_1+x_2)$ for all real numbers x_1 and x_2, which of the following could define f?

(A) $f(x)=x+1$ (B) $f(x)=2x$ (C) $f(x)=\frac{1}{x}$ (D) $f(x)=e^x$ (E) $f(x)=x^2$

20. If $y = \arctan(\cos x)$, then $\dfrac{dy}{dx} =$

(A) $\dfrac{-\sin x}{1+\cos^2 x}$ (B) $-(\operatorname{arcsec}(\cos x))^2 \sin x$ (C) $(\operatorname{arcsec}(\cos x))^2$

(D) $\dfrac{1}{(\arccos x)^2 + 1}$ (E) $\dfrac{1}{1+\cos^2 x}$

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21. If the domain of the function f given by $f(x) = \dfrac{1}{1-x^2}$ is $\{x : |x| > 1\}$, what is the range of f?

(A) $\{x : -\infty < x < -1\}$ (B) $\{x : -\infty < x < 0\}$ (C) $\{x : -\infty < x < 1\}$

(D) $\{x : -1 < x < \infty\}$ (E) $\{x : 0 < x < \infty\}$

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22. $\displaystyle\int_1^2 \frac{x^2-1}{x+1}\,dx =$

(A) $\dfrac{1}{2}$ (B) 1 (C) 2 (D) $\dfrac{5}{2}$ (E) $\ln 3$

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23. $\dfrac{d}{dx}\left(\dfrac{1}{x^3} - \dfrac{1}{x} + x^2\right)$ at $x = -1$ is

(A) -6 (B) -4 (C) 0 (D) 2 (E) 6

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24. If $\displaystyle\int_{-2}^{2} \left(x^7 + k\right) dx = 16$, then $k =$

(A) -12 (B) -4 (C) 0 (D) 4 (E) 12

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25. If $f(x) = e^x$, which of the following is equal to $f'(e)$?

(A) $\displaystyle\lim_{h\to 0} \frac{e^{x+h}}{h}$ (B) $\displaystyle\lim_{h\to 0} \frac{e^{x+h} - e^e}{h}$ (C) $\displaystyle\lim_{h\to 0} \frac{e^{e+h} - e}{h}$

(D) $\displaystyle\lim_{h\to 0} \frac{e^{x+h} - 1}{h}$ (E) $\displaystyle\lim_{h\to 0} \frac{e^{e+h} - e^e}{h}$

26. The graph of $y^2 = x^2 + 9$ is symmetric to which of the following?

I. The x-axis
II. The y-axis
III. The origin

(A) I only (B) II only (C) III only (D) I and II only (E) I, II, and III

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27. $\int_0^3 |x-1|\,dx =$

(A) 0 (B) $\frac{3}{2}$ (C) 2 (D) $\frac{5}{2}$ (E) 6

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28. If the position of a particle on the x-axis at time t is $-5t^2$, then the average velocity of the particle for $0 \le t \le 3$ is

(A) -45 (B) -30 (C) -15 (D) -10 (E) -5

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29. Which of the following functions are continuous for all real numbers x ?

I. $y = x^{\frac{2}{3}}$
II. $y = e^x$
III. $y = \tan x$

(A) None (B) I only (C) II only (D) I and II (E) I and III

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30. $\int \tan(2x)\,dx =$

(A) $-2\ln|\cos(2x)| + C$ (B) $-\frac{1}{2}\ln|\cos(2x)| + C$ (C) $\frac{1}{2}\ln|\cos(2x)| + C$

(D) $2\ln|\cos(2x)| + C$ (E) $\frac{1}{2}\sec(2x)\tan(2x) + C$

31. The volume of a cone of radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$. If the radius and the height both increase at a constant rate of $\frac{1}{2}$ centimeter per second, at what rate, in cubic centimeters per second, is the volume increasing when the height is 9 centimeters and the radius is 6 centimeters?

(A) $\frac{1}{2}\pi$ (B) 10π (C) 24π (D) 54π (E) 108π

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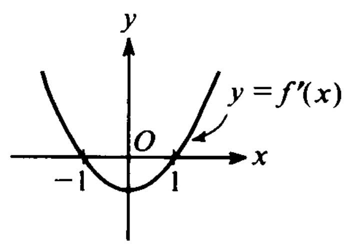
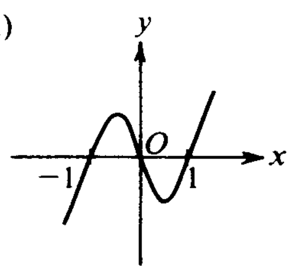
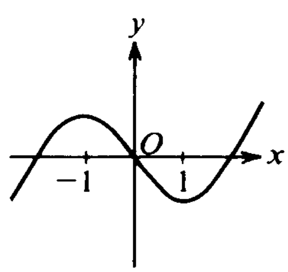
32. $\int_0^{\frac{\pi}{3}} \sin(3x)\,dx =$

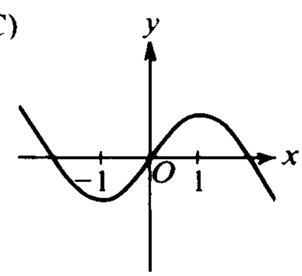
(A) -2 (B) $-\frac{2}{3}$ (C) 0 (D) $\frac{2}{3}$ (E) 2

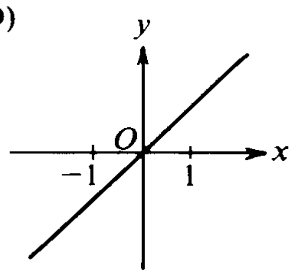


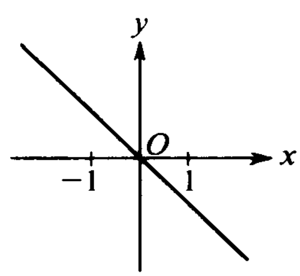
33. The graph of the derivative of f is shown in the figure above. Which of the following could be the graph of f?

(A)


(B)


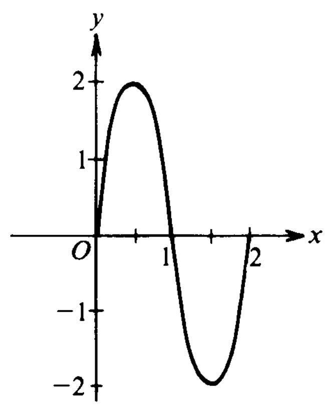
(C)


(D)


(E)


34. The area of the region in the first quadrant that is enclosed by the graphs of $y = x^3 + 8$ and $y = x + 8$ is

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) $\frac{3}{4}$ (D) 1 (E) $\frac{65}{4}$



35. The figure above shows the graph of a sine function for one complete period. Which of the following is an equation for the graph?

(A) $y = 2\sin\left(\frac{\pi}{2}x\right)$ (B) $y = \sin(\pi x)$ (C) $y = 2\sin(2x)$

(D) $y = 2\sin(\pi x)$ (E) $y = \sin(2x)$

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36. If f is a continuous function defined for all real numbers x and if the maximum value of $f(x)$ is 5 and the minimum value of $f(x)$ is -7, then which of the following must be true?

I. The maximum value of $f(|x|)$ is 5.
II. The maximum value of $|f(x)|$ is 7.
III. The minimum value of $f(|x|)$ is 0.

(A) I only (B) II only (C) I and II only (D) II and III only (E) I, II, and III

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37. $\lim_{x \to 0} (x \csc x)$ is

(A) $-\infty$ (B) -1 (C) 0 (D) 1 (E) ∞

38. Let f and g have continuous first and second derivatives everywhere. If $f(x) \le g(x)$ for all real x, which of the following must be true?

I. $f'(x) \le g'(x)$ for all real x
II. $f''(x) \le g''(x)$ for all real x
III. $\int_0^1 f(x)\,dx \le \int_0^1 g(x)\,dx$

(A) None (B) I only (C) III only (D) I and II only (E) I, II, and III

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39. If $f(x) = \dfrac{\ln x}{x}$, for all $x > 0$, which of the following is true?

(A) f is increasing for all x greater than 0.
(B) f is increasing for all x greater than 1.
(C) f is decreasing for all x between 0 and 1.
(D) f is decreasing for all x between 1 and e.
(E) f is decreasing for all x greater than e.

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40. Let f be a continuous function on the closed interval $[0, 2]$. If $2 \le f(x) \le 4$, then the greatest possible value of $\int_0^2 f(x)\,dx$ is

(A) 0 (B) 2 (C) 4 (D) 8 (E) 16

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41. If $\lim_{x \to a} f(x) = L$, where L is a real number, which of the following must be true?

(A) $f'(a)$ exists.
(B) $f(x)$ is continuous at $x = a$.
(C) $f(x)$ is defined at $x = a$.
(D) $f(a) = L$
(E) None of the above

42. $\frac{d}{dx}\int_2^x \sqrt{1+t^2}\,dt =$

(A) $\frac{x}{\sqrt{1+x^2}}$ (B) $\sqrt{1+x^2}-5$ (C) $\sqrt{1+x^2}$

(D) $\frac{x}{\sqrt{1+x^2}}-\frac{1}{\sqrt{5}}$ (E) $\frac{1}{2\sqrt{1+x^2}}-\frac{1}{2\sqrt{5}}$

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43. An equation of the line tangent to $y = x^3 + 3x^2 + 2$ at its point of inflection is

(A) $y = -6x - 6$ (B) $y = -3x + 1$ (C) $y = 2x + 10$

(D) $y = 3x - 1$ (E) $y = 4x + 1$

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44. The average value of $f(x) = x^2\sqrt{x^3+1}$ on the closed interval $[0,2]$ is

(A) $\frac{26}{9}$ (B) $\frac{13}{3}$ (C) $\frac{26}{3}$ (D) 13 (E) 26

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45. The region enclosed by the graph of $y = x^2$, the line $x = 2$, and the x-axis is revolved about the y-axis. The volume of the solid generated is

(A) 8π (B) $\frac{32}{5}\pi$ (C) $\frac{16}{3}\pi$ (D) 4π (E) $\frac{8}{3}\pi$

1985 Calculus AB Solutions

1. D $\int_1^2 x^{-3}\,dx = -\frac{1}{2}x^{-2}\Big|_1^2 = -\frac{1}{2}\left(\frac{1}{4}-1\right) = \frac{3}{8}$.

2. E $f'(x) = 4(2x+1)^3\cdot 2$, $f''(1) = 4\cdot 3(2x+1)^2\cdot 2^2$, $f'''(1) = 4\cdot 3\cdot 2(2x+1)^1\cdot 2^3$, $f^{(4)}(1) = 4!\cdot 2^4 = 384$

3. A $y = 3(4+x^2)^{-1}$ so $y' = -3(4+x^2)^{-2}(2x) = \frac{-6x}{(4+x^2)^2}$

Or using the quotient rule directly gives $y' = \frac{\left(4+x^2\right)(0)-3(2x)}{\left(4+x^2\right)^2} = \frac{-6x}{\left(4+x^2\right)^2}$

4. C $\int \cos(2x)\,dx = \frac{1}{2}\int \cos(2x)\,(2\,dx) = \frac{1}{2}\sin(2x)+C$

5. D $\lim_{n\to\infty}\frac{4n^2}{n^2+10000n} = \lim_{n\to\infty}\frac{4}{1+\frac{10000}{n}} = 4$

6. C $f'(x) = 1 \Rightarrow f'(5) = 1$

7. E $\int_1^4 \frac{1}{t}\,dt = \ln t\Big|_1^4 = \ln 4 - \ln 1 = \ln 4$

8. B $y = \ln\left(\frac{x}{2}\right) = \ln x - \ln 2$, $y' = \frac{1}{x}$, $y'(4) = \frac{1}{4}$

9. D Since e^{-x^2} is even, $\int_{-1}^0 e^{-x^2}\,dx = \frac{1}{2}\int_{-1}^1 e^{-x^2}\,dx = \frac{1}{2}k$

10. D $y' = 10^{(x^2-1)}\cdot\ln(10)\cdot\frac{d}{dx}\left((x^2-1)\right) = 2x\cdot 10^{(x^2-1)}\cdot\ln(10)$

11. B $v(t) = 2t+4 \Rightarrow a(t) = 2 \therefore a(4) = 2$

12. C $f\left(g(x)\right) = \ln\left(g(x)^2\right) = \ln\left(x^2+4\right) \Rightarrow g(x) = \sqrt{x^2+4}$

13. A $2x + x \cdot y' + y + 3y^2 \cdot y' = 0 \Rightarrow y' = -\dfrac{2x+y}{x+3y^2}$

14. D Since $v(t) \ge 0$, distance = $\int_0^4 \left| v(t) \right| dt = \int_0^4 \left(3t^{\frac{1}{2}} + 5t^{\frac{3}{2}} \right) dt = \left(2t^{\frac{3}{2}} + 2t^{\frac{5}{2}} \right) \Big|_0^4 = 80$

15. C $x^2 - 4 > 0 \Rightarrow \left| x \right| > 2$

16. B $f'(x) = 3x^2 - 6x = 3x(x-2)$ changes sign from positive to negative only at $x = 0$.

17. C Use the technique of antiderivatives by parts:

$u = x \qquad dv = e^{-x}\,dx$

$du = dx \qquad v = -e^{-x}$

$-xe^{-x} + \int e^{-x}\,dx = \left(-xe^{-x} - e^{-x} \right) \Big|_0^1 = 1 - 2e^{-1}$

18. C $y = \cos^2 x - \sin^2 x = \cos 2x$, $y' = -2\sin 2x$

19. B Quick solution: lines through the origin have this property.

Or, $f(x_1) + f(x_2) = 2x_1 + 2x_2 = 2(x_1 + x_2) = f(x_1 + x_2)$

20. A $\dfrac{dy}{dx} = \dfrac{1}{1+\cos^2 x} \cdot \dfrac{d}{dx}\left(\cos x\right) = \dfrac{-\sin x}{1+\cos^2 x}$

21. B $\left| x \right| > 1 \Rightarrow x^2 > 1 \Rightarrow f(x) < 0$ for all x in the domain. $\lim\limits_{|x| \to \infty} f(x) = 0$. $\lim\limits_{|x| \to 1} f(x) = -\infty$. The only option that is consistent with these statements is (B).

22. A $\int_1^2 \dfrac{x^2-1}{x+1}\,dx = \int_1^2 \dfrac{(x+1)(x-1)}{x+1}\,dx = \int_1^2 (x-1)\,dx = \dfrac{1}{2}(x-1)^2 \Big|_1^2 = \dfrac{1}{2}$

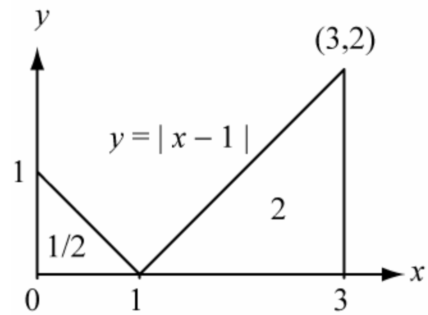
23. B $\dfrac{d}{dx}\left(x^{-3} - x^{-1} + x^2 \right) \Big|_{x=-1} = \left(-3x^{-4} + x^{-2} + 2x \right) \Big|_{x=-1} = -3 + 1 - 2 = -4$

24. D $16 = \int_{-2}^{2} (x^7 + k)\,dx = \int_{-2}^{2} x^7\,dx + \int_{-2}^{2} k\,dx = 0 + \left(2 - (-2) \right)k = 4k \Rightarrow k = 4$

25. E $f'(e)=\lim_{h\to 0}\frac{f(e+h)-f(e)}{h}=\lim_{h\to 0}\frac{e^{e+h}-e^{e}}{h}$

26. E I: Replace y with $(-y)$: $(-y)^2=x^2+9\Rightarrow y^2=x^2+9$, no change, so yes.
II: Replace x with $(-x)$: $y^2=(-x)^2+9\Rightarrow y^2=x^2+9$, no change, so yes.
III: Since there is symmetry with respect to both axes there is origin symmetry.

27. D The graph is a V with vertex at $x=1$. The integral gives the sum of the areas of the two triangles that the V forms with the horizontal axis for x from 0 to 3. These triangles have areas of 1/2 and 2 respectively.



28. C Let $x(t)=-5t^2$ be the position at time t. Average velocity $=\frac{x(3)-x(0)}{3-0}=\frac{-45-0}{3}=-15$

29. D The tangent function is not defined at $x=\pi/2$ so it cannot be continuous for all real numbers. Option E is the only one that includes item III. In fact, the functions in I and II are a power and an exponential function that are known to be continuous for all real numbers x.

30. B $\int \tan(2x)\,dx=-\frac{1}{2}\int\frac{-2\sin(2x)}{\cos(2x)}\,dx=-\frac{1}{2}\ln\left|\cos(2x)\right|+C$

31. C $V=\frac{1}{3}\pi r^2h$, $\frac{dV}{dt}=\frac{1}{3}\pi\left(2rh\frac{dr}{dt}+r^2\frac{dh}{dt}\right)=\frac{1}{3}\pi\left(2(6)(9)\left(\frac{1}{2}\right)+6^2\left(\frac{1}{2}\right)\right)=24\pi$

32. D $\int_0^{\pi/3}\sin(3x)\,dx=-\frac{1}{3}\cos(3x)\Big|_0^{\pi/3}=-\frac{1}{3}\left(\cos\pi-\cos 0\right)=\frac{2}{3}$

33. B f' changes sign from positive to negative at $x=-1$ and therefore f changes from increasing to decreasing at $x=-1$.

Or f' changes sign from positive to negative at $x=-1$ and from negative to positive at $x=1$. Therefore f has a local maximum at $x=-1$ and a local minimum at $x=1$.

34. A $\int_0^1\left((x+8)-(x^3+8)\right)dx=\int_0^1(x-x^3)\,dx=\left(\frac{1}{2}x^2-\frac{1}{4}x^4\right)\Big|_0^1=\frac{1}{4}$

35. D The amplitude is 2 and the period is 2.

$y = A\sin Bx$ where $|A| =$ amplitude $= 2$ and $B = \dfrac{2\pi}{\text{period}} = \dfrac{2\pi}{2} = \pi$

36. B II is true since $|-7| = 7$ will be the maximum value of $|f(x)|$. To see why I and III do not have to be true, consider the following: $f(x) = \begin{cases} 5 & \text{if} \quad x \le -5 \\ -x & \text{if} \quad -5 < x < 7 \\ -7 & \text{if} \quad x \ge 7 \end{cases}$

For $f(|x|)$, the maximum is 0 and the minimum is –7.

37. D $\lim\limits_{x\to 0} x\csc x = \lim\limits_{x\to 0} \dfrac{x}{\sin x} = 1$

38. C To see why I and II do not have to be true consider $f(x) = \sin x$ and $g(x) = 1 + e^x$. Then $f(x) \le g(x)$ but neither $f'(x) \le g'(x)$ nor $f''(x) < g''(x)$ is true for all real values of x.

III is true, since

$f(x) \le g(x) \Rightarrow g(x) - f(x) \ge 0 \Rightarrow \int_0^1 \big(g(x) - f(x)\big)\,dx \ge 0 \Rightarrow \int_0^1 f(x)\,dx \le \int_0^1 g(x)\,dx$

39. E $f'(x) = \dfrac{1}{x}\cdot\dfrac{1}{x} - \dfrac{1}{x^2}\ln x = \dfrac{1}{x^2}(1 - \ln x) < 0$ for $x > e$. Hence f is decreasing. for $x > e$.

40. D $\int_0^2 f(x)\,dx \le \int_0^2 4\,dx = 8$

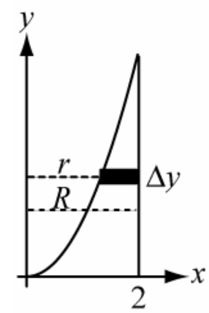
41. E Consider the function whose graph is the horizontal line $y = 2$ with a hole at $x = a$. For this function $\lim\limits_{x\to a} f(x) = 2$ and none of the given statements are true.

42. C This is a direct application of the Fundamental Theorem of Calculus: $f'(x) = \sqrt{1 + x^2}$

43. B $y' = 3x^2 + 6x$, $y'' = 6x + 6 = 0$ for $x = -1$. $y'(-1) = -3$. Only option B has a slope of –3.

44. A $\dfrac{1}{2}\int_0^2 x^2\left(x^3+1\right)^{1/2}dx = \dfrac{1}{2}\cdot\dfrac{1}{3}\int_0^2 \left(x^3+1\right)^{1/2}\left(3x^2dx\right) = \dfrac{1}{6}\left(x^3+1\right)^{3/2}\cdot\dfrac{2}{3}\Bigg|_0^2 = \dfrac{26}{9}$

45. A Washers: $\sum \pi\left(R^2 - r^2\right)\Delta y$ where $R = 2,\ r = x$



$$\text{Volume} = \pi\int_0^4 \left(2^2 - x^2\right)dy = \pi\int_0^4 (4 - y)\,dy = \pi\left(4y - \frac{1}{2}y^2\right)\Bigg|_0^4 = 8\pi$$