AP® Calculus AB
AP® Calculus BC

Free-Response Questions
and Solutions
1979 – 1988

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Notes About AP Calculus Free-Response Questions

- The solution to each free-response question is based on the scoring guidelines from the AP Reading. Where appropriate, modifications have been made by the editor to clarify the solution. Other mathematically correct solutions are possible.

- Scientific calculators were permitted, but not required, on the AP Calculus Exams in 1983 and 1984.

- Scientific (nongraphing) calculators were required on the AP Calculus Exams in 1993 and 1994.

- Graphing calculators have been required on the AP Calculus Exams since 1995. From 1995 to 1999, the calculator could be used on all six free-response questions. Since the 2000 exams, the free-response section has consisted of two parts -- Part A (questions 1-3) requires a graphing calculator and Part B (questions 4-6) does not allow the use of a calculator.

- Always refer to the most recent edition of the Course Description for AP Calculus AB and BC for the most current topic outline, as earlier exams may not reflect current exam topics.

1979 AB1

Given the function f defined by $f(x) = 2x^3 - 3x^2 - 12x + 20$.

(a) Find the zeros of f.

(b) Write an equation of the line normal to the graph of f at $x = 0$.

(c) Find the x- and y-coordinates of all points on the graph of f where the line tangent to the graph is parallel to the x-axis.

1979 AB1
Solution

(a) $f(x) = 2x^3 - 3x^2 - 12x + 20 = (x-2)(2x^2 + x - 10) = 2(x-2)^2\left(x + \frac{5}{2}\right)$

The zeros of f are at $x = 2$ and $x = -\frac{5}{2}$.

(b) $f'(x) = 6x^2 - 6x - 12;\ f'(0) = -12$

The slope of the normal line is $m = -\frac{1}{f'(0)} = \frac{1}{12};\ f(0) = 20$

The equation of the normal line is

$y - 20 = \frac{1}{12}(x - 0)$, or $y = \frac{1}{12}x + 20$, or $12y = x + 240$

(c) The tangent line will be parallel to the x-axis if the slope is 0.

$f'(x) = 0 \Rightarrow 6x^2 - 6x - 12 = 0$

$6(x-2)(x+1) = 0 \Rightarrow x = 2, -1$

$f(-1) = 27$ and $f(2) = 0$

The coordinates are $(-1, 27)$ and $(2, 0)$.

1979 AB2

A function f is defined by $f(x) = xe^{-2x}$ with domain $0 \le x \le 10$.

(a) Find all values of x for which the graph of f is increasing and all values of x for which the graph is decreasing.

(b) Give the x- and y-coordinates of all absolute maximum and minimum points on the graph of f. Justify your answers.

1979 AB2
Solution

(a) $f'(x) = e^{-2x} - 2xe^{-2x} = e^{-2x}(1-2x)$

$f'(x) > 0$ when $1-2x > 0$.

The graph of *f* is increasing for $0 \le x < \frac{1}{2}$.

$f'(x) < 0$ when $1-2x < 0$.

The graph of *f* is decreasing for $\frac{1}{2} < x \le 10$.

(b) $f'(x) = 0 \Rightarrow e^{-2x}(1-2x) = 0$

There is a critical point when $1-2x = 0$, hence only at $x = \frac{1}{2}$.

$0 < x < \frac{1}{2} \Rightarrow f'(x) > 0$

$\frac{1}{2} < x < 10 \Rightarrow f'(x) < 0$

The graph of *f* increases and then decreases on the interval $0 \le x \le 10$. Therefore the absolute maximum point is at $\left(\frac{1}{2}, \frac{1}{2e}\right)$.

The absolute minimum value must be at an endpoint.

$f(0) = 0,\ f(10) = \frac{10}{e^{20}}$

Therefore the absolute minimum point is at (0,0)

The absolute maximum can also be justified by using the second derivative test to show that there is a relative maximum at $x = \frac{1}{2}$, then observing that the absolute maximum also occurs at this *x* value since it is the only critical point in the domain.

1979 AB3/BC3

Find the maximum volume of a box that can be made by cutting out squares from the corners of an 8-inch by 15-inch rectangular sheet of cardboard and folding up the sides. Justify your answer.

1979 AB3/BC3
Solution

$$V(x) = x(8-2x)(15-2x) = 4x^3 - 46x^2 + 120x$$

$$V'(x) = 12x^2 - 92x + 120$$

$$3x^2 - 23x + 30 = (3x-5)(x-6) = 0$$
$$x = \frac{5}{3},\ x = 6$$

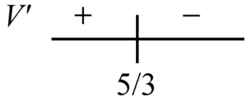
Since we must have $0 \le x \le 4$, we pick $x = \frac{5}{3}$.

$$V_{\max} = \frac{5}{3}\left(8 - \frac{10}{3}\right)\left(15 - \frac{10}{3}\right) = \frac{5}{3} \cdot \frac{14}{3} \cdot \frac{35}{3} = \frac{2450}{27} = 90\frac{20}{27} \approx 90.7$$

Justification using the 1st derivative test:

$$x < \frac{5}{3} \Rightarrow V'(x) > 0$$
$$x > \frac{5}{3} \Rightarrow V'(x) < 0$$

or



There is therefore a relative maximum at $x = \frac{5}{3}$. But since $V(0) = 0$ and $V(4) = 0$, the absolute maximum is at $x = \frac{5}{3}$.

Justification using the 2nd derivative test:

$V''\left(\frac{5}{3}\right) < 0$ and so there is a relative maximum at $x = \frac{5}{3}$. There is only one critical point in the domain $0 \le x \le 4$, so there is an absolute maximum at $x = \frac{5}{3}$.

1979 AB4/BC1

A particle moves along a line so that at any time t its position is given by $x(t) = 2\pi t + \cos 2\pi t$.

(a) Find the velocity at time t.

(b) Find the acceleration at time t.

(c) What are all values of t, $0 \le t \le 3$, for which the particle is at rest?

(d) What is the maximum velocity?

1979 AB4/BC1
Solution

(a) $v(t) = 2\pi - 2\pi\sin 2\pi t = 2\pi(1 - \sin 2\pi t)$

(b) $a(t) = -4\pi^2 \cos 2\pi t$

(c) $v(t) = 2\pi(1 - \sin 2\pi t) = 0$

$\sin 2\pi t = 1$

The particle is at rest for $t = \frac{1}{4}, \frac{5}{4}, \frac{9}{4}$.

(d) $a(t) = -4\pi^2 \cos 2\pi t = 0$

$t = \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \cdots$

The maximum velocity is $v\left(\frac{3}{4}\right) = 4\pi$.

or

Since $\sin 2\pi t = -1$ is the minimum of $\sin 2\pi t$, the maximum of $v(t)$ is $2\pi\big(1 - (-1)\big) = 4\pi$.

1979 AB5/BC5

Let R be the region bounded by the graph of $y = \frac{1}{x} \ln x$, the x-axis, and the line $x = e$.

(a) Find the area of the region R.

(b) Find the volume of the solid formed by revolving the region R about the y-axis.

1979 AB5/BC5
Solution

(a) $\frac{1}{x}\ln x = 0 \Rightarrow x = 1$

Area $= \int_1^e \frac{1}{x}\ln x\,dx = \frac{1}{2}\left(\ln x\right)^2\Big|_1^e = \frac{1}{2}$

(b) $\text{Volume} = 2\pi\int_1^e x\left(\frac{1}{x}\ln x\right)dx = 2\pi\int_1^e \ln x\,dx$

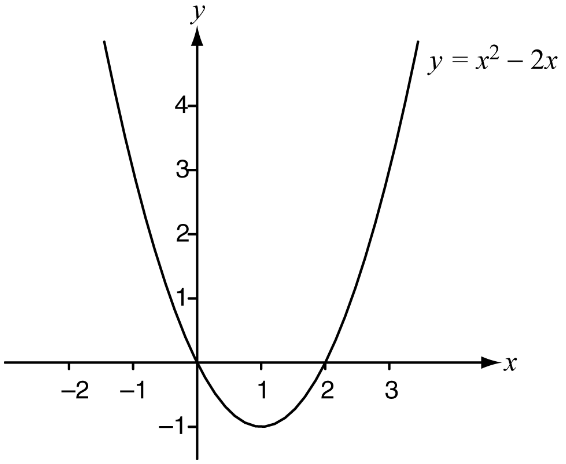
Use integration by parts with

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}dx \qquad v = x$

Volume $= 2\pi\left(x\ln x\Big|_1^e - \int_1^e 1\,dx\right) = 2\pi(x\ln x - x)\Big|_1^e = 2\pi$

1979 AB6



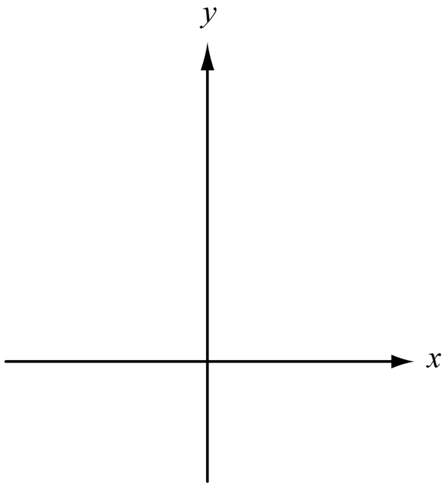
The curve in the figure represents the graph of f, where $f(x) = x^2 - 2x$ for all real numbers x.

(a) On the axes provided, sketch the graph of $y = |f(x)|$.

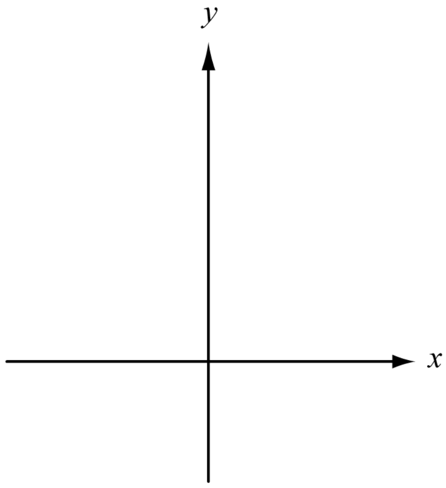
(b) Determine whether the derivative of $|f(x)|$ exists at $x = 0$. Justify your answer.

(c) On the axes provided, sketch the graph of $y = f(|x|)$.

(d) Determine whether $y = f(|x|)$ is continuous at $x = 0$. Justify your answer.



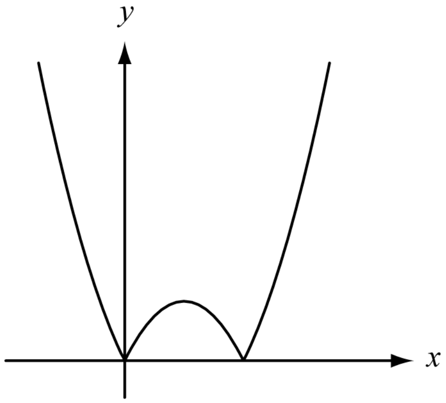
Axes for (a)



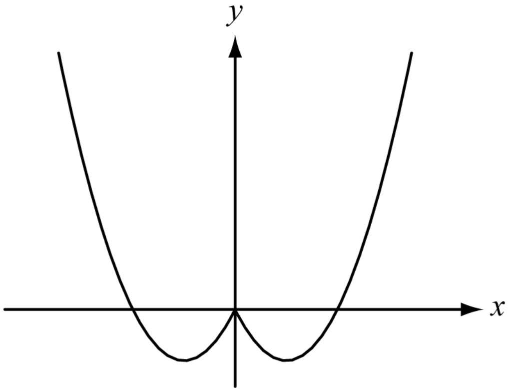
Axes for (c)

1979 AB6
Solution

(a) $y = \left| f(x) \right|$



(c) $y = f\left(\left| x \right| \right)$



(b) $x < 0 \Rightarrow \left| f(x) \right| = x^2 - 2x \Rightarrow \dfrac{d}{dx}\left| f(x) \right| = 2x - 2 \Rightarrow \lim\limits_{x \to 0^-} \dfrac{d}{dx}\left| f(x) \right| = -2$

$0 < x < 2 \Rightarrow \left| f(x) \right| = -x^2 + 2x \Rightarrow \dfrac{d}{dx}\left| f(x) \right| = -2x + 2 \Rightarrow \lim\limits_{x \to 0^+} \dfrac{d}{dx}\left| f(x) \right| = 2$

If $\left| f(x) \right|$ were differentiable at $x = 0$, then since both limits above exist, they would have to be equal. They are not equal, so the derivative of $\left| f(x) \right|$ does not exist at 0.

Alternatively, the derivative of $\left| f(x) \right|$ does not exist at $x = 0$ because

$$\lim_{h \to 0^-} \frac{\left| f(0+h) \right| - \left| f(0) \right|}{h} = \lim_{h \to 0^-} \frac{h^2 - 2h}{h} = \lim_{h \to 0^-} (h - 2) = -2$$

$$\lim_{h \to 0^+} \frac{\left| f(0+h) \right| - \left| f(0) \right|}{h} = \lim_{x \to 0^-} \frac{-h^2 + 2h}{h} = \lim_{h \to 0^-} (-h + 2) = 2$$

(d) At $x = 0, f\left(\left| x \right| \right) = 0$

$\lim\limits_{x \to 0} f\left(\left| x \right| \right) = \lim\limits_{x \to 0} \left(\left| x \right|^2 - 2\left| x \right| \right) = 0$

Therefore $f\left(\left| x \right| \right)$ is continuous at $x = 0$.

or At $x = 0, f\left(\left| x \right| \right) = 0$

$\lim\limits_{x \to 0^+} f\left(\left| x \right| \right) = \lim\limits_{x \to 0^+} f(x) = 0$

$\lim\limits_{x \to 0^-} f\left(\left| x \right| \right) = \lim\limits_{x \to 0^-} f(-x) = 0$

$\lim\limits_{x \to 0^+} f\left(\left| x \right| \right) = \lim\limits_{x \to 0^-} f\left(\left| x \right| \right) = 0$

Therefore $f\left(\left| x \right| \right)$ is continuous at $x = 0$.

1979 AB7

Let f be the function defined by $y = f(x) = x^3 + ax^2 + bx + c$ and having the following properties.

(i) The graph of f has a point of inflection at $(0, -2)$.

(ii) The average (mean) value of $f(x)$ on the closed interval $[0, 2]$ is -3.

(a) Determine the values of a, b, and c.

(b) Determine the value of x that satisfies the conclusion of the Mean Value Theorem for f on the closed interval $[0, 3]$.

1979 AB7
Solution

(a) $-2 = f(0) = c$

$$f'(x) = 3x^2 + 2ax + b$$

$$f''(x) = 6x + 2a$$

$$0 = f''(0) = 2a, \text{ so } a = 0$$

$$f(x) = x^3 + bx - 2$$

$$-3 = \frac{1}{2-0}\int_0^2 f(x)\,dx = \frac{1}{2}\left(\frac{x^4}{4} + \frac{bx^2}{2} - 2x\right)\Bigg|_0^2 = \frac{1}{2}\left(4 + 2b - 4\right) = b$$

So $a = 0,\ b = -3,$ and $c = -2$.

(b) By the Mean Value Theorem, there is an x satisfying $0 < x < 3$ such that

$$f'(x) = \frac{f(3) - f(0)}{3 - 0}$$

$$3x^2 - 3 = \frac{16 - (-2)}{3} = 6$$

$$x^2 = 3 \Rightarrow x = \sqrt{3}$$

1979 BC2

Given the differential equation $py'' + y' - 2y = qx$

(a) Find the general solution of the differential equation when $p = 0$ and $q = 0$.

(b) Find the general solution of the differential equation when $p = 1$ and $q = 0$.

(c) Find the general solution of the differential equation when $p = 1$ and $q = 2$.

1979 BC2
Solution

(a) $p = 0,\ q = 0$

$y' - 2y = 0$

$\dfrac{dy}{y} = 2\,dx$

$\ln|y| = 2x + \ln C$

$y = e^{2x + \ln C}$

The general solution is $y = Ce^{2x}$.

(b) $p = 1,\ q = 0$

$y'' + y' - 2y = 0$

$m^2 + m - 2 = 0$

$(m+2)(m-1) = 0$

$m = -2, 1$

The general solution is $y = C_1 e^{-2x} + C_2 e^{x}$.

(c) $p = 1,\ q = 2$

$y'' + y' - 2y = 2x$

$y_h = C_1 e^{-2x} + C_2 e^{x}$ from (b)

Let $y_p = Ax + B$. Then $y_p' = A$ and $y_p'' = 0$.

$0 + A - 2Ax - 2B = 2x$

$$\begin{cases} A - 2B = 0 \\ \quad -2A = 2 \end{cases}$$

The solution is $A = -1,\ B = -\dfrac{1}{2}$

Hence the general solution is $y = y_h + y_p = C_1 e^{-2x} + C_2 e^{x} - x - \dfrac{1}{2}$.

1979 BC4

Let f be the function defined by $f(x)=\dfrac{1}{1-2x}$.

(a) Write the first four terms and the general term of the Taylor series expansion of $f(x)$ about $x=0$.

(b) What is the interval of convergence for the series found in part (a)? Show your method.

(c) Find the value of f at $x=-\dfrac{1}{4}$. How many terms of the series are adequate for approximating $f\left(-\dfrac{1}{4}\right)$ with an error not exceeding one per cent? Justify your answer.

1979 BC4
Solutions

(a) The Taylor series has the form $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n$

$n = 0: f(x) = \frac{1}{1-2x};\ f(0) = 1 \qquad 1$

$n = 1: f'(x) = \frac{2}{(1-2x)^2};\ f'(0) = 2 \qquad 2x$

$n = 2: f''(x) = \frac{8}{(1-2x)^3}; f''(0) = 8 \qquad 4x^2$

$n = 3;\ f'''(x) = \frac{48}{(1-2x)^4};\ f'''(0) = 48 \qquad 8x^3$

The general term is $2^n x^n$.

(b) $\left|\frac{2^{n+1}x^{n+1}}{2^n x^n}\right| = |2x|$ so series converges for $|x| < \frac{1}{2}$

$x = \frac{1}{2}$: $\sum_{n=0}^{\infty}(2^n)\left(\frac{1}{2^n}\right) = \sum_{n=0}^{\infty}(1)^n$; divergent

$x = -\frac{1}{2}$: $\sum_{n=0}^{\infty}(2^n)\left(\frac{1}{-2}\right)^n = \sum_{n=0}^{\infty}(-1)^n$; divergent

Interval of convergence $-\frac{1}{2} < x < \frac{1}{2}$

(c) $f\left(-\frac{1}{4}\right) = \frac{1}{1+1/2} = \frac{2}{3}$.

$\sum 2^n\left(\frac{-1}{4}\right)^n = \sum(-1)^n \frac{1}{2^n}$

Alternating series with absolute value of terms a_n decreasing to 0.

$$\left|S - \left(\sum_{j=0}^{n-1} a_j\right)\right| < |a_n|$$

$$= \frac{1}{2^n} \le \frac{2}{300}$$

or

$$\left|S - \left(\sum_{j=0}^{n-1} a_j\right)\right| \le \left|\frac{f^{(n)}(c)}{n!}\left(-\frac{1}{4}\right)^n\right| \quad \text{for } -\frac{1}{4} < c < 0$$

$$= \left|\frac{(-1)^n n!(1-2c)^{-n-1}(-2)^n}{n!}\left(\frac{1}{4}\right)^n\right|$$

$$\le \frac{2^n}{4^n}\left|(1-2c)^{-n-1}\right|$$

$$\le \frac{2^n}{4^n} = \frac{1}{2^n} \le \frac{2}{300}$$

Since $\frac{1}{256} < \frac{2}{300}$, 8 terms suffice.

1979 BC6

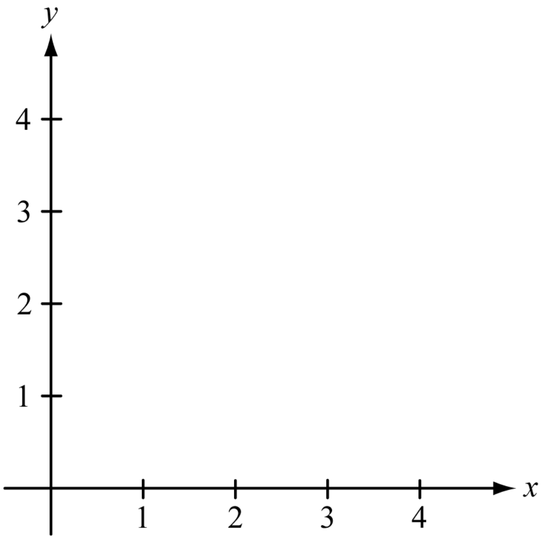
A particle moves in the xy-plane so that at any time $t \geq 0$ its position (x, y) is given by $x = e^t + e^{-t}$ and $y = e^t - e^{-t}$.

(a) Find the velocity vector for any $t \geq 0$.

(b) Find $\lim_{t \to \infty} \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

(c) The particle moves on a hyperbola. Find an equation for this hyperbola in terms of x and y.

(d) On the axes provided, sketch the path of the particle showing the velocity vector for $t = 0$.



1979 BC6
Solution

(a) $\dfrac{dx}{dt} = e^t - e^{-t} \qquad \dfrac{dy}{dt} = e^t + e^{-t}$

$$\vec{v}(t) = (e^t - e^{-t}, e^t + e^{-t}) = (e^t - e^{-t})\vec{i} + (e^t + e^{-t})\vec{j}$$

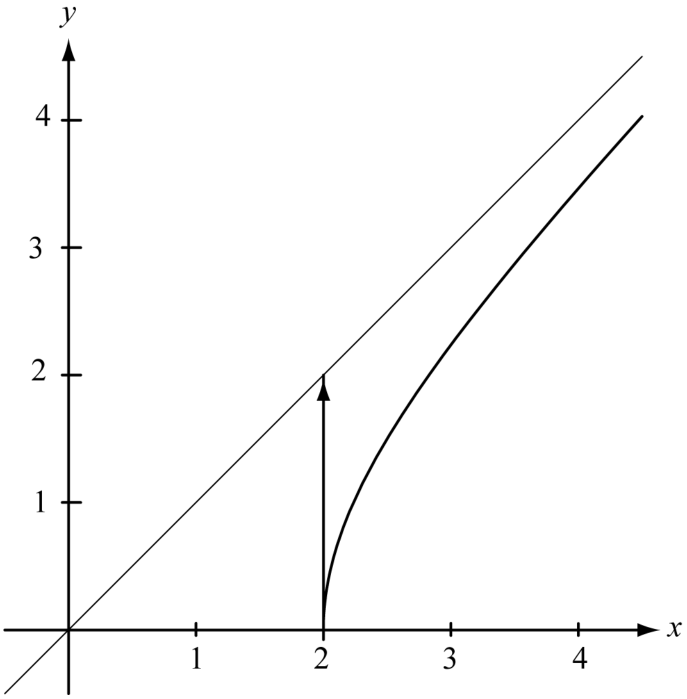
(b)

Method 1	Method 2	Method 3	Method 4
$\lim\limits_{t\to\infty} \dfrac{1+\dfrac{1}{e^{2t}}}{1-\dfrac{1}{e^{2t}}} = 1$	$\lim\limits_{t\to\infty}\left(1+\dfrac{2e^{-t}}{e^t - e^{-t}}\right) = 1$	$\lim\limits_{t\to\infty}\dfrac{e^{2t}+1}{e^{2t}-1}$ $= \lim\limits_{t\to\infty}\dfrac{2e^{2t}}{2e^{2t}} = 1$	$\lim\limits_{t\to\infty}\coth(t) = 1$

(c) $\begin{cases} x^2 = e^{2t} + 2 + e^{-2t} \\ y^2 = e^{2t} - 2 + e^{2t} \end{cases}$ or $\begin{cases} \dfrac{1}{2}x = \cosh x \\ \dfrac{1}{2}y = \sinh x \end{cases}$

Therefore $x^2 - y^2 = 4$

(d) $\vec{v}(0) = 2\vec{j} = (0,2)$



1979 BC7

Let *f* be a function with domain the set of all real numbers and having the following properties.

(i) $f(x+y)=f(x)f(y)$ for all real numbers x and y.

(ii) $\lim\limits_{h\to 0}\dfrac{f(h)-1}{h}=k$, where k is a nonzero real number.

(a) Use these properties and a definition of the derivative to show that $f'(x)$ exists for all real numbers x.

(b) Let $f^{(n)}$ denote the nth derivative of *f*. Write an expression for $f^{(n)}(x)$ in terms of $f(x)$.

(c) Given that $f(1)=2$, use the Mean Value Theorem to show that there exists a number c such that $0<c<3$ and $f'(c)=\dfrac{7}{3}$.

1979 BC7
Solution

(a) $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{f(x)f(h)-f(x)}{h}=\lim_{h\to 0}f(x)\frac{f(h)-1}{h}=k\,f(x)$

(b) $f''(x)=k\,f'(x)=k^2 f(x)$

By induction, $f^{(n)}(x)=k^n f(x)$

(c) Property (i) gives $f(1)=f(0+1)=f(0)f(1)$

Therefore $f(0)=1$

By Property (i), $f(2)=f(1)f(1)=4$

By Property (i), $f(3)=f(1)f(2)=8$

By the Mean Value Theorem, there is a c satisfying $0<c<3$ such that

$$f'(c)=\frac{f(3)-f(0)}{3-0}=\frac{8-1}{3}=\frac{7}{3}$$

1980 AB1

Let R be the region enclosed by the graphs of $y = x^3$ and $y = \sqrt{x}$.

(a) Find the area of R.

(b) Find the volume of the solid generated by revolving R about the x-axis.

1980 AB1
Solution

(a) The intersection of the two graphs is at (0, 0) and (1, 1).

$$\text{Area} = \int_0^1 (x^{1/2} - x^3)\,dx = \left(\frac{2}{3}x^{3/2} - \frac{1}{4}x^4\right)\Bigg|_0^1$$

$$= \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$$

or

$$\text{Area} = \int_0^1 \left(y^{1/3} - y^2\right)dy = \left(\frac{3}{4}y^{4/3} - \frac{1}{3}y^3\right)\Bigg|_0^1$$

$$= \frac{3}{4} - \frac{1}{3} = \frac{5}{12}$$

(b) $$\text{Volume} = \pi\int_0^1 (x - x^6)\,dx = \pi\left(\frac{1}{2}x^2 - \frac{1}{7}x^7\right)\Bigg|_0^1$$

$$= \pi\left(\frac{1}{2} - \frac{1}{7}\right) = \frac{5}{14}\pi$$

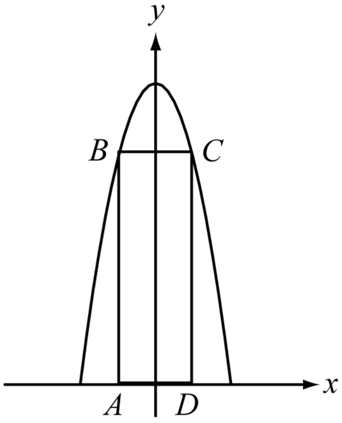
or

$$\text{Volume} = 2\pi\int_0^1 \left(y^{1/3} - y^2\right)y\,dy = 2\pi\int_0^1 \left(y^{4/3} - y^3\right)dy$$

$$= 2\pi\left(\frac{3}{7}y^{7/3} - \frac{1}{4}y^4\right)\Bigg|_0^1$$

$$= 2\pi\left(\frac{3}{7} - \frac{1}{4}\right) = \frac{5}{14}\pi$$

1980 AB2



A rectangle $ABCD$ with sides parallel to the coordinate axes is inscribed in the region enclosed by the graph of $y = -4x^2 + 4$ and the x-axis as shown in the figure above.

(a) Find the x- and y-coordinates of C so that the area of rectangle $ABCD$ is a maximum.

(b) The point C moves along the curve with its x-coordinate increasing at the constant rate of 2 units per second. Find the rate of change of the area of rectangle $ABCD$ when $x = \frac{1}{2}$.

1980 AB2
Solution

(a) $A(x) = 2x(-4x^2 + 4) = 8(x - x^3)$

$\frac{dA}{dx} = 8(1 - 3x^2)$

$\frac{dA}{dx} = 0$ when $x = \frac{1}{\sqrt{3}}$

The maximum area occurs when $x = \frac{1}{\sqrt{3}}$ and $y = 4\left(1 - \frac{1}{3}\right) = \frac{8}{3}$.

(b) $A(x) = 8(x - x^3)$

$\frac{dA}{dt} = 8(1 - 3x^2)\frac{dx}{dt}$

When $x = \frac{1}{2}$ and $\frac{dx}{dt} = 2$, $\frac{dA}{dt} = 8\left(1 - 3 \cdot \frac{1}{4}\right)2 = 4$.

or

$A = 2xy$

$\frac{dA}{dt} = 2\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right)$

When $x = \frac{1}{2}$ and $\frac{dx}{dt} = 2$,

$y = -4x^2 + 4 = -4\left(\frac{1}{2}\right)^2 + 4 = 3$

$\frac{dy}{dt} = -8x\frac{dx}{dt} = -8 \cdot \frac{1}{2} \cdot 2 = -8$

$\frac{dA}{dt} = 2\left(\frac{1}{2}(-8) + 3 \cdot 2\right) = 4$

1980 AB3

Let $\ln(x^2)$ for $x > 0$ and $g(x) = e^{2x}$ for $x \geq 0$. Let H be the composition of *f* with g, that is, $H(x) = f(g(x))$, and let K be the composition of g with *f*, that is $K(x) = g(f(x))$.

(a) Find the domain of H and write an expression for $H(x)$ that does not contain the exponential function.

(b) Find the domain of K and write an expression for $K(x)$ that does not contain the exponential function.

(c) Find an expression for $f^{-1}(x)$, where f^{-1} denotes the inverse function of *f*, and find the domain of f^{-1}.

1980 AB3
Solution

(a) The domain of H consists of x for which $x \geq 0$ and $g(x) = e^{2x} > 0$. Hence the domain is $x \geq 0$.

$$H(x) = f(g(x)) = \ln((e^{2x})^2) = \ln(e^{4x}) = 4x \text{ for } x \geq 0$$

(b) The domain of K consists of x for which $x > 0$ and $f(x) = \ln(x^2) \geq 0$. Hence the domain is $x \geq 1$.

$$K(x) = g(f(x)) = e^{2\ln(x^2)} = e^{\ln x^4} = x^4 \text{ for } x \geq 1$$

(c) $y = \ln x^2 \Rightarrow e^y = x^2$

$$\Rightarrow x = \sqrt{e^y} = e^{y/2}$$

$$\Rightarrow f^{-1}(x) = e^{x/2}$$

The domain of f^{-1} is the range of f which is the set of all real numbers.

1980 AB4/BC1

The acceleration of a particle moving along a straight line is given by $a = 10e^{2t}$.

(a) Write an expression for the velocity v, in terms of time t, if $v = 5$ when $t = 0$.

(b) During the time that the velocity increases from 5 to 15, how far does the particle travel?

(c) Write an expression for the position s, in terms of time t, of the particle if $s = 0$ when $t = 0$.

1980 AB4/BC1
Solution

(a) $a = 10e^{2t}$

$v = \int 10e^{2t}\,dt = 5e^{2t} + C$

$v(0) = 5 \Rightarrow C = 0$

Therefore $v = 5e^{2t}$

(b) $v = 5$ when $t = 0$

$v = 15 \Rightarrow 5e^{2t} = 15$

$\Rightarrow t = \frac{1}{2}\ln 3$

Distance $= \int_0^{\frac{1}{2}\ln 3} 5e^{2t}dt = \frac{5}{2}e^{2t}\Big|_0^{\frac{1}{2}\ln 3}$

$= \frac{5}{2}e^{\ln 3} - \frac{5}{2} = 5$

or

$s = \int 5e^{2t}dt = \frac{5}{2}e^{2t} + C$

Distance $= s\left(\frac{1}{2}\ln 3\right) - s(0) = \left(\frac{5}{2}e^{\ln 3} + C\right) - \left(\frac{5}{2} + C\right) = \frac{5}{2}e^{\ln 3} - \frac{5}{2} = 5$

(c) $s = \int 5e^{2t}dt = \frac{5}{2}e^{2t} + C$

$s(0) = 0 \;\Rightarrow\; C = -\frac{5}{2}$

Therefore $s = \frac{5}{2}e^{2t} - \frac{5}{2}$

1980 AB5/BC2

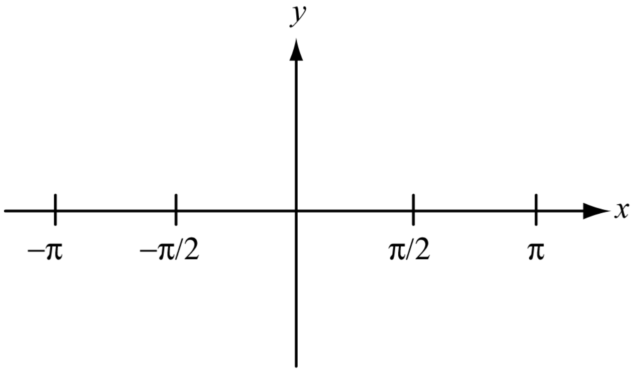
Given the function f defined by $f(x) = \cos x - \cos^2 x$ for $-\pi \le x \le \pi$.

(a) Find the x-intercepts of the graph of f.

(b) Find the x- and y-coordinates of all relative maximum points of f. Justify your answer.

(c) Find the intervals on which the graph of f is increasing.

(d) Using the information found in parts (a), (b), and (c), sketch the graph of f on the axes provided.



1980 AB5/BC2
Solution

(a) $f(x) = \cos x \cdot (1 - \cos x)$

Either $\cos x = 0$ or $1 - \cos x = 0$, so the x-intercepts are $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$, and $x = 0$.

(b) $f'(x) = -\sin x + 2\sin x \cos x$
$0 = \sin x \cdot (-1 + 2\cos x)$

Either $\sin x = 0$ or $\cos x = \frac{1}{2}$, so the candidates are $x = \pm\pi$, $x = 0$, and $x = \pm\frac{\pi}{3}$.

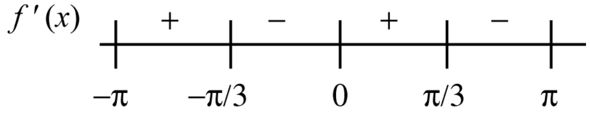
The relative maximum points are at $\left(\pm\frac{\pi}{3}, \frac{1}{4}\right)$.

Justification:
(i) $f''(x) = -\cos x + 2\cos 2x$
$f''(\pm\pi) = 3 \Rightarrow$ relative minimum
$f''(0) = 1 \Rightarrow$ relative minimum
$f''\left(\pm\frac{\pi}{3}\right) = -\frac{3}{2} \Rightarrow$ relative maximum

or
(ii) Selecting critical values:

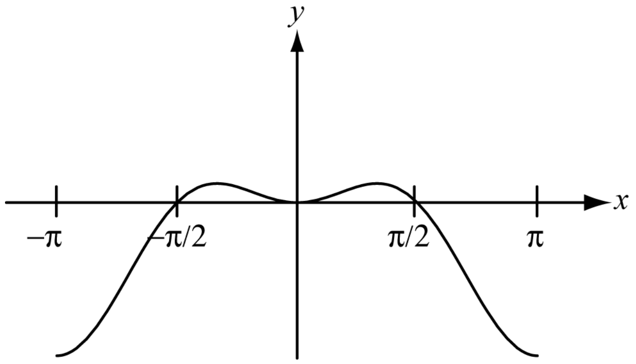
x	$-\pi$	$-\frac{\pi}{3}$	0	$\frac{\pi}{3}$	π
$f(x)$	-2	$1/4$	0	$1/4$	-2

or
(iii) Sign chart:



(c) Graph of f increases on the intervals $-\pi < x < -\frac{\pi}{3}$ and $0 < x < \frac{\pi}{3}$.

(d)



1980 AB6/BC4

Let $y = f(x)$ be the continuous function that satisfies the equation $x^4 - 5x^2y^2 + 4y^4 = 0$ and whose graph contains the points $(2,1)$ and $(-2,-2)$. Let ℓ be the line tangent to the graph of f at $x = 2$.

(a) Find an expression for y'.

(b) Write an equation for line ℓ.

(c) Give the coordinates of a point that is on the graph of f but is not on line ℓ.

(d) Give the coordinates of a point that is on line ℓ but is not on the graph of f.

1980 AB6/BC4
Solution

Solution 1:

(a) $4x^3 - 10xy^2 - 10x^2yy' + 16y^3y' = 0$

$$y' = \frac{4x^3 - 10xy^2}{10x^2y - 16y^3} = \frac{2x^3 - 5xy^2}{5x^2y - 8y^3}$$

(b) The slope is the value of y' at the point $(2,1)$, so $m = \frac{16-10}{20-8} = \frac{1}{2}$.

The equation of ℓ is therefore $y - 1 = \frac{1}{2}(x-2)$ or $y = \frac{1}{2}x$.

(c) The point $(-2,-2)$ is one example. Any point of the form (a,a) for $a < 0$ will be on the graph of f but not on the line ℓ. (For reason, see solution 2.)

(d) The point $(-2,-1)$ is one example. Any point of the form $\left(a, \frac{a}{2}\right)$ for $a < 0$ will be on the line ℓ but not on the graph of f. (For reason, see solution 2.)

Solution 2:

(a) The equation can be rewritten as $(x-2y)(x+2y)(x-y)(x+y) = 0$. Four different lines passing through the origin satisfy this implicit equation. Because $y = f(x)$ is continuous, only one line can be used for $x < 0$ and $x \geq 0$, respectively. Which line is used is determined by the two points that are given as being on the graph. So we must have

$$y = \begin{cases} \frac{1}{2}x, & x \geq 0 \\ x, & x < 0 \end{cases}$$

$$y' = \begin{cases} \frac{1}{2}, & x > 0 \\ 1, & x < 0 \end{cases}$$

(b) $y = \frac{1}{2}x$

(c) (a,a) for $a < 0$

(d) $\left(a, \frac{a}{2}\right)$ for $a < 0$

1980 AB7

Let p and q be real numbers and let f be the function defined by:

$$f(x)=\begin{cases}1+2p(x-1)+(x-1)^2, & \text{for } x\le 1\\ qx+p, & \text{for } x>1.\end{cases}$$

(a) Find the value of q, in terms of p, for which f is continuous at $x=1$.

(b) Find the values of p and q for which f is differentiable at $x=1$.

(c) If p and q have the values determined in part (b), is f'' a continuous function? Justify your answer.

1980 AB7
Solution

(a) Must have $f(1) = \lim_{x \to 1} f(x)$ and $f(1) = 1$

$\lim_{x \to 1^+} f(x) = p + q$ and $\lim_{x \to 1^-} f(x) = 1$

Therefore $p + q = 1$

$q = 1 - p$

(b) $\lim_{x \to 1^-} f'(x) = 2p$

$\lim_{x \to 1^+} f'(x) = q$

So for $f'(1)$ to exist, $2p = q$

$f'(1)$ exists implies that f is continuous at $x = 1$. Therefore $q = 1 - p$.

Hence $2p = 1 - p$.

$p = \frac{1}{3}$, $q = \frac{2}{3}$.

(c) <u>No</u>, f'' is not a continuous function because it is not continuous at $x = 1$. This is because f'' is not defined at $x = 1$, or because $\lim_{x \to 1^-} f''(x) = 2$ and $\lim_{x \to 1^+} f''(x) = 0$.

or

<u>Yes</u>, f'' is a continuous function because f'' is continuous at each point of its domain.

(Note: Different answers were accepted on the 1980 grading standard because students might have interpreted the question either as asking if f'' is a continuous function for all real numbers, or a continuous function on its domain.)

1980 BC3

(a) Determine whether the series $A = \sum_{n=1}^{\infty} \frac{4n}{n^2+1}$ converges or diverges. Justify your answer.

(b) If S is the series formed by multiplying the nth term in A by the nth term in $\sum_{n=1}^{\infty} \frac{1}{2n}$, write an expression using summation notation for S.

(c) Determine whether the series S found in part (b) converges or diverges. Justify your answer.

1980 BC3
Solution

(a) Comparison or limit comparison test

$$\frac{4n}{n^2+1} > \frac{1}{n} \text{ since } 4n^2 > n^2+1 \qquad \text{or} \qquad \lim_{n\to\infty}\left(\frac{4n}{n^2+1} \div \frac{1}{n}\right) = \lim_{n\to\infty}\frac{4n^2}{n^2+1} = 4$$

Since the series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, so does series *A*.

Integral test

$$\lim_{b\to\infty}\int_1^b \frac{4x}{x^2+1}\,dx = \lim_{b\to\infty} 2\ln(x^2+1)\Big|_1^b = \infty$$

Since the integral diverges, so does series *A*.

(b) $\sum_{n=1}^{\infty}\frac{2}{n^2+1}$

(c) Comparison or limit comparison test

$$\frac{2}{n^2+1} < \frac{2}{n^2} \qquad \text{or} \qquad \lim_{n\to\infty}\left(\frac{2}{n^2+1} \div \frac{2}{n^2}\right) = \lim_{n\to\infty}\frac{n^2}{n^2+1} = 1$$

Since the series $\sum_{n=1}^{\infty}\frac{2}{n^2}$ converges, so does series *S*.

Integral test

$$\lim_{b\to\infty}\int_1^b \frac{2}{x^2+1}\,dx = \lim_{b\to\infty} 2\tan^{-1}x\Big|_1^b = \frac{\pi}{2}$$

Since the integral converges, so does series *S*.

1980 BC5

(a) Find the general solution of the differential equation $xy' + y = 0$.

(b) Find the general solution of the differential equation $xy' + y = 2x^2 y$.

(c) Find the particular solution of the differential equation in part (b) that satisfies the condition that $y = e^2$ when $x = 1$.

1980 BC5
Solution

(a) $xy' + y = 0$

Solution using separation of variables

$$\frac{y'}{y} = \frac{-1}{x} \quad \Rightarrow \quad \ln y = -\ln x + C$$

Solution using integrating factor

$$y' + \frac{1}{x}y = 0, \quad P(x) = \frac{1}{x}$$

$$y = Ae^{-\int \frac{1}{x}dx}$$

In either case, $y = Ae^{-\ln x}$ or $y = \dfrac{A}{x}$ or $xy = A$ or $\ln|y| = -\ln|x| + C$

(b) $xy' + y = 2x^2 y$

Solution using separation of variables

$$\frac{y'}{y} = \frac{2x^2 - 1}{x} = 2x - \frac{1}{x}$$

$$\ln y = x^2 - \ln x + C$$

Solution using integrating factor

$$y' + \left(\frac{1 - 2x^2}{x}\right)y = 0, \quad P(x) = \frac{1}{x} - 2x$$

$$y = Ae^{-\int\left(\frac{1}{x} - 2x\right)dx}$$

In either case, $y = \dfrac{Ae^{x^2}}{x}$ or $y = Ae^{x^2 - \ln x}$ or $\ln|y| = x^2 - \ln|x| + C$

(c) $e^2 = Ae$

$e = A$

Therefore $y = \dfrac{e \cdot e^{x^2}}{x} = \dfrac{e^{x^2+1}}{x}$ or $\ln y = x^2 - \ln x + 1$

1980 BC6

Let R be the region enclosed by the graphs of $y = e^{-x}$, $x = k\ (k > 0)$, and the coordinate axes.

(a) Write an improper integral that represents the limit of the area of the region R as k increases without bound and find the value of the integral if it exists.

(b) Find the volume, in terms of k, of the solid generated if R is rotated about the y-axis.

(c) Find the volume, in terms of k, of the solid whose base is R and whose cross sections perpendicular to the x-axis are squares.

1980 BC6
Solution

(a) Area = $\int_0^\infty e^{-x}\,dx = \lim_{k\to\infty}\int_0^k e^{-x}\,dx$

$$= \lim_{k\to\infty}(-e^{-x})\Big|_0^k$$

$$= \lim_{k\to\infty}(-e^{-k}+1) = 1$$

(b) Volume = $\int_0^k 2\pi x e^{-x}\,dx \qquad (u = x;\ v' = e^{-x})$

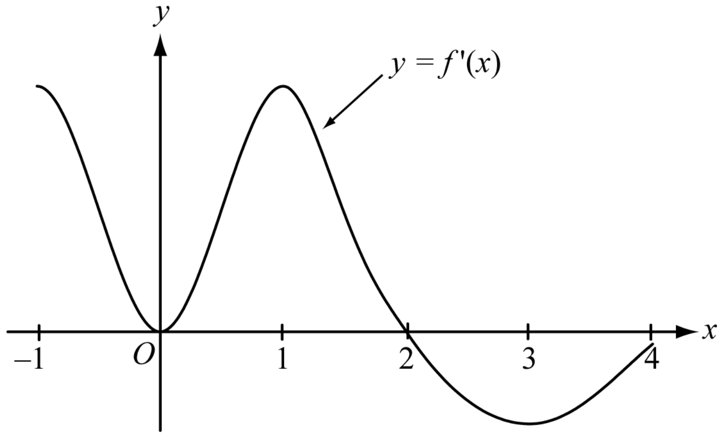
$$= 2\pi\left(-xe^{-x}\Big|_0^k - \int_0^k -e^{-x}\,dx\right)$$

$$= 2\pi(-xe^{-x} - e^{-x})\Big|_0^k$$

$$= 2\pi(-ke^{-k} - e^{-k} + 1)$$

(c) Volume = $\int_0^k (e^{-x})^2\,dx = \int_0^k e^{-2x}\,dx = -\frac{1}{2}e^{-2x}\Big|_0^k = -\frac{1}{2}(e^{-2k}-1)$

1980 BC7



<u>Note</u>: This is the graph of the <u>derivative</u> of *f*, NOT the graph of *f*.

Let *f* be a function that has domain the closed interval $[-1,4]$ and range the closed interval $[-1,2]$. Let $f(-1)=-1$, $f(0)=0$, and $f(4)=1$. Also let *f* have the derivative function f' that is continuous and that has the graph shown in the figure above.

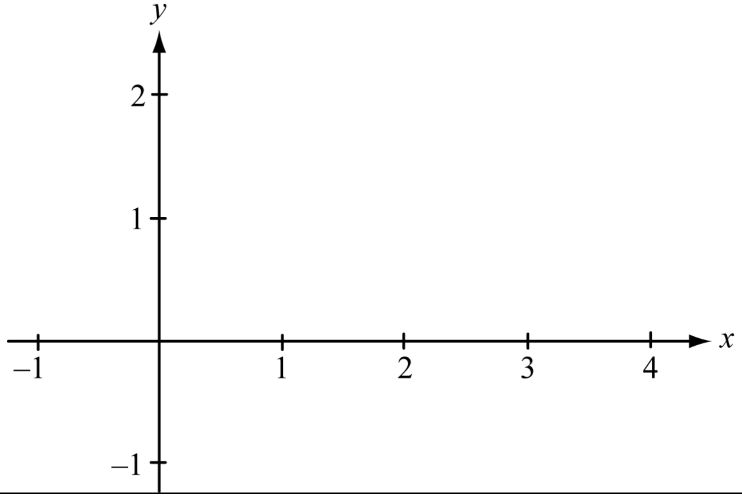
(a) Find all values of x for which *f* assumes a relative maximum. Justify your answer.

(b) Find all values of x for which *f* assumes its absolute minimum. Justify your answer.

(c) Find the intervals on which *f* is concave downward.

(d) Give all the values of x for which *f* has a point of inflection.

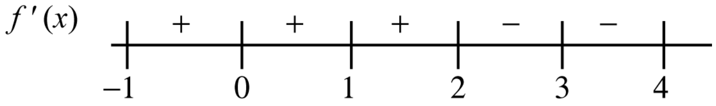
(e) On the axes provided, sketch the graph of *f*.



Note: The graph of f' has been slightly modified from the original on the 1980 exam to be consistent with the given values of f at $x=-1$, $x=0$, and $x=4$.

1980 BC7
Solution

(a) $f'(x) = 0$ at $x = 0, 2$



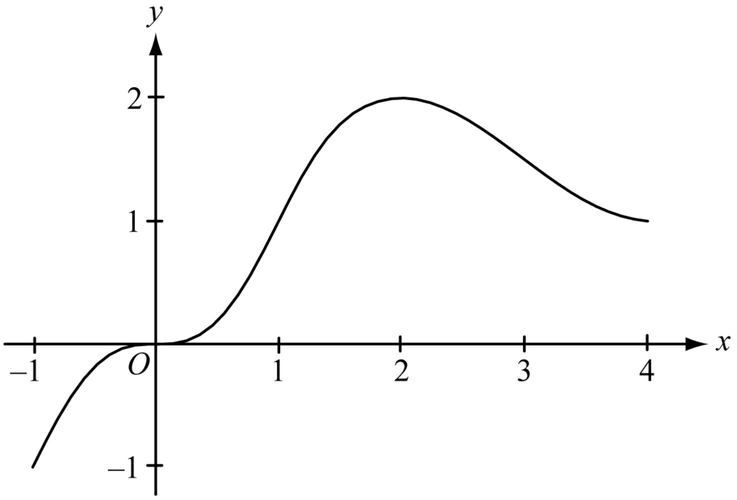
There is a relative maximum at $x = 2$, since $f'(2) = 0$ and $f'(x)$ changes from positive to negative at $x = 2$.

(b) There is no minimum at $x = 0$, since $f'(x)$ does not change sign there. So the absolute minimum must occur at an endpoint. Since $f(-1) < f(4)$, the absolute minimum occurs at $x = -1$.

(c) The graph of f is concave down on the intervals $[-1, 0)$ and $(1, 3)$ because f' is decreasing on those intervals.

(d) The graph of f has a point of inflection at $x = 0$, 1, and 3 because f' changes from decreasing to increasing or from increasing to decreasing at each of those x values.

(e)



1981 AB1

Let f be the function defined by $f(x) = x^4 - 3x^2 + 2$.

(a) Find the zeros of f.

(b) Write an equation of the line tangent to the graph of f at the point where $x = 1$.

(c) Find the x-coordinate of each point at which the line tangent to the graph of f is parallel to the line $y = -2x + 4$.

1981 AB1
Solution

(a) $f(x) = (x^2 - 1)(x^2 - 2) = (x+1)(x-1)(x+\sqrt{2})(x-\sqrt{2})$

The zeros are $x = \pm 1, \pm\sqrt{2}$.

(b) $f'(x) = 4x^3 - 6x$
$f'(1) = -2$
Point: (1,0)

The equation of the tangent line is $y = -2(x-1)$.

(c) $4x^3 - 6x = -2$
$4x^3 - 6x + 2 = 0$
One solution is $x = 1$.
$(x-1)(4x^2 + 4x - 2) = 0$
The other two solutions are $x = \frac{1}{2}\left(-1 \pm \sqrt{3}\right)$.

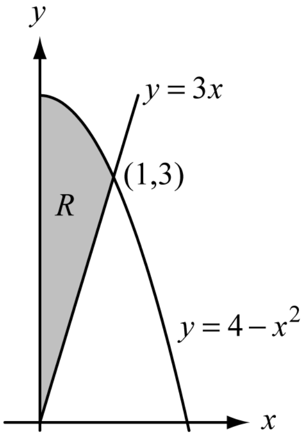
1981 AB2

Let R be the region in the first quadrant enclosed by the graphs of $y = 4 - x^2$, $y = 3x$, and the y-axis.

(a) Find the area of region R.

(b) Find the volume of the solid formed by revolving the region R about the x-axis.

1981 AB2
Solution



(a) Intersection:

$4 - x^2 = 3x$
$x^2 + 3x - 4 = 0$
$x = -4, x = 1$

Area = $\int_0^1 \left(4 - x^2 - 3x\right) dx = \left(4x - \frac{1}{3}x^3 - \frac{3}{2}x^2\right)\Bigg|_0^1 = \frac{13}{6}$

or

Area = $\int_0^3 \frac{1}{3} y\,dy + \int_0^1 \left(4 - y\right)^{1/2} dy = \frac{13}{6}$

(b) Disks: $\text{Volume} = \pi\int_0^1 \left(\left(4 - x^2\right)^2 - \left(3x\right)^2\right) dx$

$$= \pi\int_0^1 \left(16 - 17x^2 + x^4\right) dx$$

$$= \pi\left(16x - \frac{17}{3}x^3 + \frac{1}{5}x^5\right)\Bigg|_0^1 = \frac{158\pi}{15}$$

or

Shells: $\text{Volume} = 2\pi\int_0^3 y \cdot \frac{1}{3} y\,dy + 2\pi\int_3^4 y\left(4 - y\right)^{1/2} dy$

$$= 2\pi \cdot \frac{y^3}{9}\Bigg|_0^3 + 2\pi\left(-\frac{2}{3}y\left(4 - y\right)^{3/2} - \frac{4}{15}\left(4 - y\right)^{5/2}\right)\Bigg|_3^4 = \frac{158\pi}{15}$$

1981 AB3/BC1

Let f be the function defined by $f(x) = 12x^{\frac{2}{3}} - 4x$.

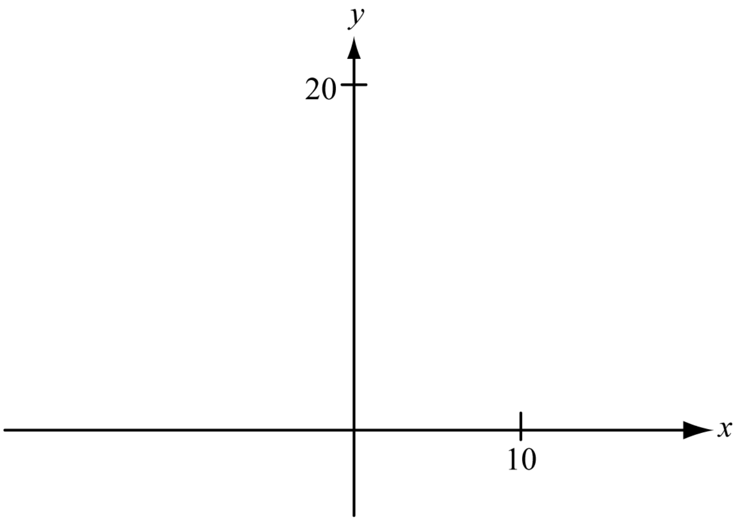
(a) Find the intervals on which f is increasing.

(b) Find the x- and y-coordinates of all relative maximum points.

(c) Find the x- and y-coordinates of all relative minimum points.

(d) Find the intervals on which f is concave downward.

(e) Using the information found in parts (a), (b), (c), and (d), sketch the graph of f on the axes provided.



1981 AB3/BC1
Solution

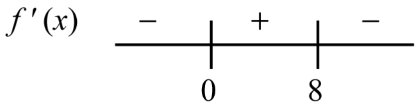
(a) $f(x) = 12x^{2/3} - 4x;\ f'(x) = 8x^{-1/3} - 4$

$(8x^{-1/3} - 4) > 0,\ x > 0 \Rightarrow x < 8$

$(8x^{-1/3} - 4) > 0,\ x < 0 \Rightarrow$ no x satisfies this

or

Critical numbers: $x = 8$, $x = 0$



Therefore f is increasing on the interval $0 < x < 8$.

(b) 2nd Derivative Test: $f''(x) = -\frac{8}{3}x^{-4/3}$

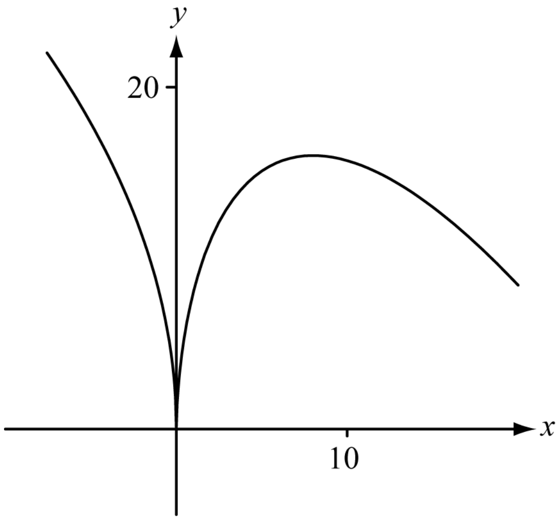
$f''(8) < 0 \Rightarrow$ relative maximum at (8,16)

(c) The 2nd Derivative Test cannot be used at $x = 0$ where the second derivative is undefined. Since $f'(x) < 0$ for x just less than 0, and $f'(x) > 0$ for x just greater than 0, there is a relative minimum at $(0,0)$.

(d) $f''(x) = -\frac{8}{3}x^{-4/3} < 0$ if $x \neq 0$

The graph of f is concave down on $(-\infty, 0)$ and $(0, +\infty)$.

(e)



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1981 AB4

Let *f* be the function defined by $f(x) = 5^{\sqrt{2x^2-1}}$.

(a) Is *f* an even or odd function? Justify your answer.

(b) Find the domain of *f*.

(c) Find the range of *f*.

(d) Find $f'(x)$.

1981 AB4
Solution

(a) f is even since

$$f(-x)=5^{\sqrt{2(-x)^2-1}}=5^{\sqrt{2x^2-1}}=f(x)$$

(b) $2x^2-1\ge 0$

$$x^2\ge\frac{1}{2}$$

Domain is $\left(-\infty,\frac{-1}{\sqrt{2}}\right]\cup\left[\frac{1}{\sqrt{2}},\infty\right)$

(c) $\sqrt{2x^2-1}\ge 0\Rightarrow 5^{\sqrt{2x^2-1}}\ge 1$

Range is $[1,\infty)$

(d) $$f'(x)=5^{\sqrt{2x^2-1}}\cdot\ln 5\cdot\frac{1}{2\sqrt{2x^2-1}}\cdot 4x$$

or

$$y=5^{\sqrt{2x^2-1}}$$

$$\ln y=\sqrt{2x^2-1}\cdot\ln 5$$

$$\frac{1}{y}y'=\frac{1}{2\sqrt{2x^2-1}}\cdot 4x\cdot\ln 5$$

$$y'=\frac{y}{2\sqrt{2x^2-1}}\cdot 4x\cdot\ln 5$$

$$y'=\frac{5^{\sqrt{2x^2-1}}}{2\sqrt{2x^2-1}}\cdot 4x\cdot\ln 5$$

1981 AB5/BC2

Let f be the function defined by $f(x)=\begin{cases} 2x+1, & \text{for } x\le 2 \\ \frac{1}{2}x^2+k, & \text{for } x>2 \end{cases}$

(a) For what value of k will f be continuous at $x=2$? Justify your answer.

(b) Using the value of k found in part (a), determine whether f is differentiable at $x=2$. Use the definition of the derivative to justify your answer.

(c) Let $k=4$. Determine whether f is differentiable at $x=2$. Justify your answer.

1981 AB5/BC2
Solution

(a) $f(2) = 5$

$\lim\limits_{x \to 2^-} (2x+1) = 5$

$\lim\limits_{x \to 2^+} \left(\frac{1}{2}x^2 + k \right) = 2 + k$

For continuity at $x = 2$, we must have $2 + k = 5$, and so $k = 3$.

(b) We compute $\lim\limits_{x \to 2} \frac{f(x) - f(2)}{x - 2}$:

$\lim\limits_{x \to 2^-} \frac{2x+1-5}{x-2} = 2$

$\lim\limits_{x \to 2^+} \frac{\frac{1}{2}x^2 + 3 - 5}{x-2} = 2$

So $f'(2)$ exists and $f'(2) = 2$

(c) When $k = 4$,

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} \left(\frac{1}{2}x^2 + 4 \right) = 6$

Hence f is not continuous at $x = 2$ and so is not differentiable at $x = 2$.

1981 AB6/BC4

A particle moves along the x-axis so that at time t its position is given by $x(t) = \sin\left(\pi t^2\right)$ for $-1 \le t \le 1$.

(a) Find the velocity at time t.

(b) Find the acceleration at time t.

(c) For what values of t does the particle change direction?

(d) Find all values of t for which the particle is moving to the left.

1981 AB6/BC4
Solution

(a) $v(t) = x'(t) = 2\pi t \cos \pi t^2$

(b) $a(t) = v'(t) = 2\pi \cos \pi t^2 - 4\pi^2 t^2 \sin \pi t^2$

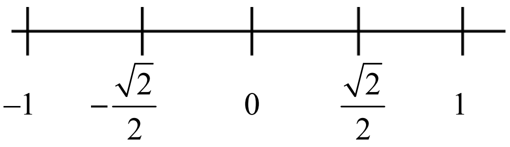
(c) $v(t) = 0$

$t = 0$ or $\cos \pi t^2 = 0$

$\pi t^2 = \pm \frac{\pi}{2}$

The particle changes direction at $t = \pm \frac{\sqrt{2}}{2}, 0$.

(d) The particle is moving to the left when $v(t) < 0$.



t	$-$	$-$	$+$	$+$
$\cos \pi t^2$	$-$	$+$	$+$	$-$
$v(t)$	$+$	$-$	$+$	$-$

Particle moves to the left when $-\frac{\sqrt{2}}{2} < t < 0$ or $\frac{\sqrt{2}}{2} < t < 1$.

1981 AB7

Let f be a continuous function that is defined for all real numbers x and that has the following properties.

(i) $\int_1^3 f(x)\,dx = \frac{5}{2}$ (ii) $\int_1^5 f(x)\,dx = 10$

(a) Find the average (mean) value of f over the closed interval $[1,3]$.

(b) Find the value of $\int_3^5 \left(2f(x)+6\right)dx$.

(c) Given that $f(x) = ax + b$, find the values of a and b.

1981 AB7
Solution

(a) Average (mean) value $= \frac{1}{3-1}\int_1^3 f(x)\,dx = \frac{1}{2}\left(\frac{5}{2}\right) = \frac{5}{4}$

(b) $\int_3^5 \left(2f(x)+6\right)dx$

$= 2\int_3^5 f(x)\,dx + \int_3^5 6\,dx$

$= 2\left(\int_1^5 f(x)\,dx - \int_1^3 f(x)\,dx\right) + \int_3^5 6\,dx$

$= 2\left(10 - \frac{5}{2}\right) + 6(5-3)$

$= 15 + 12 = 27$

(c) $\frac{5}{2} = \int_1^3 (ax+b)dx = \left(\frac{ax^2}{2} + bx\right)\Big|_1^3 = 4a + 2b$

$10 = \int_1^5 (ax+b)dx = \left(\frac{ax^2}{2} + bx\right)\Big|_1^5 = 12a + 4b$

Solving these two simultaneous equations yields $a = \frac{5}{4},\ b = -\frac{5}{4}$.

1981 BC3

Let S be the series $S = \sum_{n=0}^{\infty} \left(\frac{t}{1+t} \right)^n$ where $t \neq 0$.

(a) Find the value to which S converges when $t = 1$.

(b) Determine the values of t for which S converges. Justify your answer.

(c) Find all the values of t that make the sum of the series S greater than 10.

1981 BC3
Solution

(a) $t = 1 \Rightarrow \dfrac{t}{1+t} = \dfrac{1}{2}$

Then $S = \displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} = 2$

(b) Since *S* is a geometric series, *S* converges if and only if $|r| = \left|\dfrac{t}{1+t}\right| < 1$.

(i) We must have $|t| < |t+1|$. This means that the distance of *t* from 0 is less than the distance of *t* from –1. Therefore $t > -\dfrac{1}{2}$.

or

(ii) We must have $t^2 < t^2 + 2t + 1$. Therefore $t > -\dfrac{1}{2}$.

or

(iii) If $t < -1$: $t + 1 < t < -1 - t$
$t + 1 < t \Rightarrow 1 < 0$
no solution
If $t > -1$: $-t - 1 < t < t + 1$
$-1 < 2t$
$-\dfrac{1}{2} < t$
Therefore $t > -\dfrac{1}{2}$.

(c) $S(t) = \dfrac{1}{1 - \frac{t}{1+t}} = 1 + t > 10$ for $t > 9$

1981 BC5

(a) Find the general solution of the differential equation $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}-6y=0$.

(b) Find the particular solution for the differential equation in part (a) that satisfies the conditions that $y=1$ and $\dfrac{dy}{dx}=-1$ when $x=0$.

(c) Find the general solution of the differential equation $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}-6y=e^x$.

1981 BC5
Solution

(a) $m^2 + m - 6 = 0$

$(m+3)(m-2) = 0$

$m = -3, 2$

$y = Ae^{2x} + Be^{-3x}$

(b) $y' = 2Ae^{2x} - 3Be^{-3x}$

$y = 1$ when $x = 0$ gives $A + B = 1$

$y' = -1$ when $x = 0$ gives $2A - 3B = -1$

Therefore $A = \frac{2}{5}$ and $B = \frac{3}{5}$.

$y = \frac{2}{5}e^{2x} + \frac{3}{5}e^{-3x}$

(c) From part (a), the homogeneous solution is $y_h = Ae^{2x} + Be^{-3x}$.

Try a particular solution of the form $y_p = Ce^x$.

$y_p = y_p{}' = y_p{}'' = Ce^x$

$Ce^x + Ce^x - 6Ce^x = e^x$

$-4C = 1$

$C = -\frac{1}{4}$

Therefore the general solution is $y = Ae^{2x} + Be^{-3x} - \frac{1}{4}e^x$.

1981 BC6

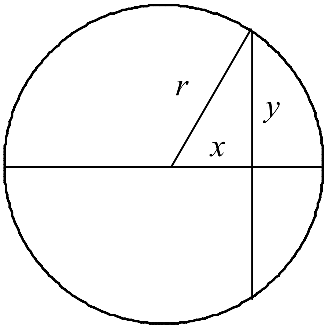
(a) A solid is constructed so that it has a circular base of radius r centimeters and every plane section perpendicular to a certain diameter of the base is a square, with a side of the square being a chord of the circle. Find the volume of the solid.

(b) If the solid described in part (a) expands so that the radius of the base increases at a constant rate of $\frac{1}{2}$ centimeters per minute, how fast is the volume changing when the radius is 4 centimeters?

1981 BC6
Solution

(a) The cross section at x has area

$A(x) = (2y)^2 = 4\left(r^2 - x^2\right)$.



$$\text{Volume } = \int_{-r}^{r} 4\left(r^2 - x^2\right)dx$$
$$= 2\int_0^r 4\left(r^2 - x^2\right)dx$$
$$= 8\left(r^2 x - \frac{1}{3}x^3\right)\Bigg|_0^r = \frac{16}{3}r^3$$

(b) $V = \dfrac{16}{3}r^3$

$$\frac{dV}{dt} = 16r^2\frac{dr}{dt} = 16(4^2)\frac{1}{2} = 128$$

or

$\dfrac{dr}{dt} = \dfrac{1}{2}$ for all t implies that $r = \dfrac{1}{2}t + C$.

Choose $t = 0$ when $r = 0$. Then $C = 0$ and

$$V = \frac{16}{3}r^3 = \frac{16}{3}\left(\frac{1}{2}t\right)^3 = \frac{2}{3}t^3$$
$$\frac{dV}{dt} = 2t^2$$

When $r = 4$, then $t = 8$ and $\dfrac{dV}{dt} = 2\cdot 8^2 = 128$.

1981 BC7

Let f be a differentiable function defined for all $x > 0$ such that

(i) $f(1) = 0$,
(ii) $f'(1) = 1$, and
(iii) $\frac{d}{dx}\left[f(2x)\right] = f'(x)$, for all $x > 0$.

(a) Find $f'(2)$.

(b) Suppose f' is differentiable. Prove that there is a number c, $2 < c < 4$, such that $f''(c) = -\frac{1}{8}$.

(c) Prove that $f(2x) = f(2) + f(x)$ for all $x > 0$.

1981 BC7
Solution

(a) $\frac{d}{dx}\left[f(2x)\right]=f'(x)$ by (iii)

$f'(2x)\cdot 2=f'(x)$

$f'(2x)=\frac{1}{2}f'(x)$ (*)

$f'(2)=\frac{1}{2}f'(1)=\frac{1}{2}$ by (ii)

(b) There is a c, $2<c<4$, so that $f''(c)=\frac{f'(4)-f'(2)}{4-2}$ by the Mean Value theorem.

$f'(2)=\frac{1}{2}$ from (a)

$f'(4)=\frac{1}{4}$ from (*)

Therefore $f''(c)=\frac{\frac{1}{4}-\frac{1}{2}}{4-2}=-\frac{1}{8}$.

(c) $\frac{d}{dx}\left[f(2x)\right]=\frac{d}{dx}f(x)$ by (iii)

Therefore $f(2x)=f(x)+C$

$f(2)=f(1)+C=C$ by (i)

$f(2x)=f(x)+f(2)$

1982 AB1

A particle moves along the x-axis in such a way that its acceleration at time t for $t > 0$ is given by $a(t) = \frac{3}{t^2}$. When $t = 1$, the position of the particle is 6 and the velocity is 2.

(a) Write an equation for the velocity, $v(t)$, of the particle for all $t > 0$.

(b) Write an equation for the position, $x(t)$, of the particle for all $t > 0$.

(c) Find the position of the particle when $t = e$.

1982 AB1
Solution

(a) $v(t) = \int a(t)\,dt = \int \frac{3}{t^2}\,dt = -\frac{3}{t} + C$

$2 = v(1) = -3 + C$

Therefore $C = 5$ and so $v(t) = -\frac{3}{t} + 5$.

(b) $x(t) = \int v(t)\,dt = \int (-\frac{3}{t} + 5)dt = -3\ln t + 5t + C$

$6 = x(1) = -3\ln 1 + 5 + C$

Therefore $C = 1$ and so $x(t) = -3\ln t + 5t + 1$.

(c) $x(e) = -3\ln e + 5e + 1 = -3 + 5e + 1 = 5e - 2$

1982 AB2

Given that f is the function defined by $f(x) = \dfrac{x^3 - x}{x^3 - 4x}$.

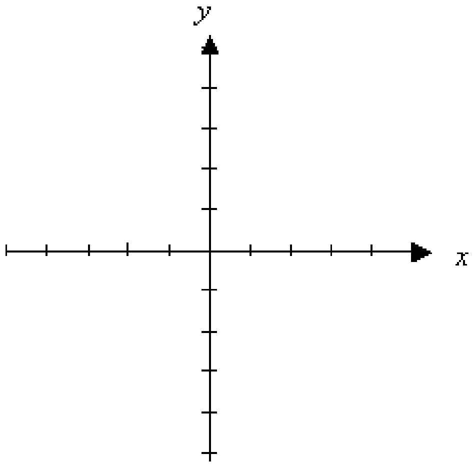
(a) Find the $\lim\limits_{x \to 0} f(x)$.

(b) Find the zeros of f.

(c) Write an equation for each vertical and each horizontal asymptote to the graph of f.

(d) Describe the symmetry of the graph of f.

(e) Using the information found in parts (a), (b), (c), and (d), sketch the graph of f on the axes provided.



1982 AB2
Solution

(a) $\lim\limits_{x\to 0}\dfrac{x^3-x}{x^3-4x}=\lim\limits_{x\to 0}\dfrac{x^2-1}{x^2-4}=\dfrac{1}{4}$

(b) $f(x)=0$ for $x^3-x=0,\ x\neq 0$

$x^2-1=0$

The zeros are $x=1$ and $x=-1$.

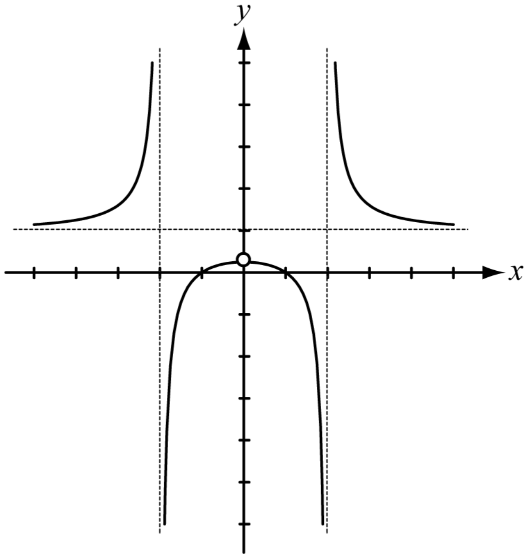
(c) Vertical asymptote: $x=2, x=-2$

Horiztonal asymptote: $y=1$

(d) The graph is symmetric with respect to the y-axis.

(because $f(-x)=\dfrac{(-x)^3-(-x)}{(-x)^3-4(-x)}=\dfrac{-x^3+x}{-x^3+4x}=\dfrac{x^3-x}{x^3-4x}=f(x)$)

(e)



1982 AB3/BC1

Let R be the region in the first quadrant that is enclosed by the graph of $y = \tan x$, the x-axis, and the line $x = \frac{\pi}{3}$.

(a) Find the area of R.

(b) Find the volume of the solid formed by revolving R about the x-axis.

1982 AB3/BC1
Solution

(a) $$\text{Area} = \int_0^{\pi/3} \tan x\, dx$$

$$= -\ln(\cos x)\Big|_0^{\pi/3}$$

$$= -\ln\frac{1}{2} = \ln 2$$

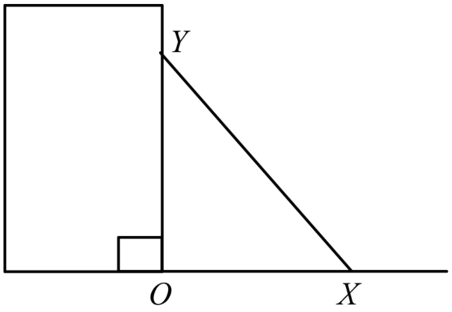
(b) $$\text{Volume} = \int_0^{\pi/3} \pi \tan^2 x\, dx$$

$$= \pi\int_0^{\pi/3} (\sec^2 x - 1)\, dx$$

$$= \pi(\tan x - x)\Big|_0^{\pi/3}$$

$$= \pi\left(\sqrt{3} - \frac{\pi}{3}\right)$$

1982 AB4



A ladder 15 feet long is leaning against a building so that end X is on level ground and end Y is on the wall as shown in the figure. X is moved away from the building at the constant rate of $\frac{1}{2}$ foot per second.

(a) Find the rate in feet per second at which the length OY is changing when X is 9 feet from the building.

(b) Find the rate of change in square feet per second of the area of triangle XOY when X is 9 feet from the building.

1982 AB4
Solution

(a) $x^2 + y^2 = 15^2$

Implicit:	Explicit:
$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$	$y = \left(15^2 - x^2\right)^{1/2}$
$9 \cdot \frac{1}{2} + 12\frac{dy}{dt} = 0$	$\frac{dy}{dt} = \frac{-x}{\left(15^2 - x^2\right)^{1/2}}\frac{dx}{dt}$
$\frac{dy}{dt} = -\frac{3}{8}$	$\frac{dy}{dt} = \frac{-9\left(\frac{1}{2}\right)}{12} = -\frac{3}{8}$

(b) $A = \frac{1}{2}xy$

Implicit:	Explicit:
$\frac{dA}{dt} = \frac{1}{2}\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right)$	$A = \frac{1}{2}x\left(15^2 - x^2\right)^{1/2}$
$\frac{dA}{dt} = \frac{1}{2}\left(9 \cdot \left(-\frac{3}{8}\right) + 12 \cdot \frac{1}{2}\right)$	$\frac{dA}{dt} = \frac{1}{2}\frac{15^2 - 2x^2}{\left(15^2 - x^2\right)^{1/2}}\frac{dx}{dt}$
$\frac{dA}{dt} = \frac{21}{16}$	$\frac{dA}{dt} = \frac{21}{16}$

1982 AB5/BC2

Let f be the function defined by $f(x) = (x^2 + 1)\, e^{-x}$ for $-4 \le x \le 4$.

(a) For what value of x does f reach its absolute maximum? Justify your answer.

(b) Find the x-coordinates of all points of inflection of f. Justify your answer.

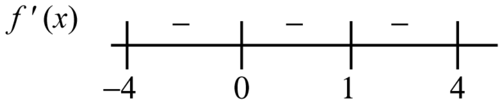
1982 AB5/BC2
Solution

(a) $f(x) = (x^2 + 1)e^{-x} \qquad -4 \le x \le 4$

$f'(x) = 2xe^{-x} - (x^2 + 1)e^{-x} = -e^{-x}(x+1)^2$

$f'(x) \le 0$ for all x and therefore f is decreasing for all x.

or



Since f is decreasing on the entire interval, the absolute maximum is at $x = –4$.

or

The absolute maximum is at a critical point or an endpoint. There is a critical point at $x = 1$.

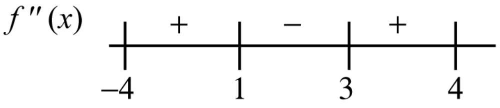
$f(–4) = 17e^4$

$f(1) = \dfrac{2}{e}$

$f(4) = \dfrac{17}{e^4}$

Therefore the absolute maximum is at $x = –4$.

(b) $f''(x) = e^{-x}(x-1)^2 - e^{-x} \cdot 2(x-1) = e^{-x}(x-1)(x-3)$

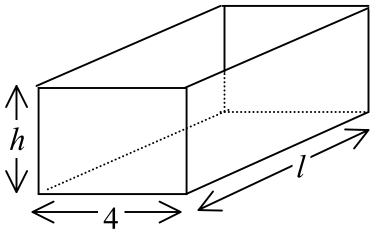


$f''(x) > 0 \qquad -4 < x < 1$
$f''(x) < 0 \qquad 1 < x < 3$
$f''(x) > 0 \qquad 3 < x < 4$
The points of inflection are at $x = 1$ and $x = 3$.

1982 AB6/BC3

A tank with a rectangular base and rectangular sides is to be open at the top. It is to be constructed so that its width is 4 meters and its volume is 36 cubic meters. If building the tank costs $10 per square meter for the base and $5 per square meter for the sides, what is the cost of the least expensive tank?

1982 AB6/BC3
Solution



$$\text{Volume} = 4lh = 36$$
$$\text{Therefore } lh = 9$$
$$\text{Cost} = C = 10(4l) + 5\big(2(4h) + 2hl\big) = 40l + 40h + 10hl$$

Method 1: Direct solution

$$C = 40\left(\frac{9}{h}\right) + 40h + 90$$
$$C' = 40\left(-\frac{9}{h^2} + 1\right)$$
$$C' = 0 \text{ when } h = \pm 3$$
$$\text{Thus } \ h = 3 \text{ and } l = 3$$

Method 2: Implicit Differentiation

$$\frac{dl}{dh} = -\frac{l}{h}$$
$$\frac{dC}{dh} = 40\frac{dl}{dh} + 40$$
$$\frac{dC}{dh} = 40\left(-\frac{l}{h}\right) + 40$$
$$-1 = -\frac{l}{h}$$
$$l = h = 3$$

When $l = 3$ and $h = 3$, then $C = 40\big(3\big) + 40\big(3\big) + 90 = \$330$.

1982 AB7

For all real numbers x, f is a differentiable function such that $f(-x) = f(x)$.
Let $f(p) = 1$ and $f'(p) = 5$ for some p>0.

(a) Find $f'(-p)$.

(b) Find $f'(0)$.

(c) If ℓ_1 and ℓ_2 are lines tangent to the graph of f at $(-p, 1)$ and $(p, 1)$, respectively, and if ℓ_1 and ℓ_2 intersect at point Q, find the x- and y-coordinates of Q in terms of p.

1982 AB7
Solution

(a) $f(-x) = f(x)$
$\Rightarrow -f'(-x) = f'(x)$
$\Rightarrow f'(-p) = -f'(p) = -5$

(b) $f'(-0) = -f'(0) \Rightarrow f'(0) = 0$

(c) Equations of the tangent lines
$\ell_1 : y - 1 = -5(x + p)$
$\ell_2 : y - 1 = 5(x - p)$

Solution 1:
At the intersection, we must have $-5x - 5p = 5x - 5p$ and so $x = 0$. The coordinates of Q are $x = 0,\ y = 1 - 5p$.

Solution 2:
Since f is even, the tangent lines ℓ_1 and ℓ_2 intersect on the y-axis since they are tangent at symmetric points on the graph with respect to the y-axis. Therefore $x = 0$ at the point of intersection. The y-coordinate is $y = 1 - 5p$.

1982 BC4

A particle moves along the x-axis so that its position function $x(t)$ satisfies the differential equation $\frac{d^2x}{dt^2} - \frac{dx}{dt} - 6x = 0$ and has the property that at time $t = 0,\ x = 2,$ and $\frac{dx}{dt} = -9.$

(a) Write an expression for $x(t)$ in terms of t.

(b) At what times $t,$ if any, does the particle pass through the origin?

(c) At what times $t,$ if any, is the particle at rest?

1982 BC4
Solution

(a) $\dfrac{d^2x}{dt^2} - \dfrac{dx}{dt} - 6x = 0$

$$r^2 - r - 6 = 0$$
$$r = -2, 3$$
$$x = C_1 e^{-2t} + C_2 e^{3t}$$
$$\frac{dx}{dt} = -2C_1 e^{-2t} + 3C_2 e^{3t}$$

$$2 = C_1 + C_2$$
$$-9 = -2C_1 + 3C_2$$
$$C_1 = 3,\ C_2 = -1$$

$$x(t) = 3e^{-2t} - e^{3t}$$

(b) The particle passes through the origin when $x(t) = 0$.

$$3e^{-2t} = e^{3t}$$
$$3 = e^{5t}$$
$$t = \frac{1}{5}\ln 3$$

(c) The particle is at rest when $\dfrac{dx}{dt} = 0$.

$$\frac{dx}{dt} = -6e^{-2t} - 3e^{3t} = -3(2e^{-2t} + e^{3t}) < 0$$

Since $\dfrac{dx}{dt} < 0$ for all t, the particle is never at rest.

1982 BC5

(a) Write the Taylor series expansion about $x = 0$ for $f(x) = \ln(1+x)$. Include an expression for the general term.

(b) For what values of x does the series in part (a) converge?

(c) Estimate the error in evaluating $\ln\left(\frac{3}{2}\right)$ by using only the first five nonzero terms of the series in part (a). Justify your answer.

(d) Use the result found in part (a) to determine the logarithmic function whose Taylor series is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n}}{2n}$.

1982 BC5
Solution

(a) $f(x) = \ln(1+x)$ $\qquad f(0) = 0$

$f'(x) = (1-x)^{-1}$ $\qquad f'(0) = 1$

$f''(x) = -(1-x)^{-2}$ $\qquad f''(0) = -1$

$\cdots$ $\qquad \cdots$

$f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n}$ $\qquad f^{(n)}(0) = (-1)^{n-1}(n-1)!$

$$f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots$$

$$= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} \text{ or } \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$$

(b) The radius of convergence is $R = 1$.

at $x = 1$: $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ converges

at $x = -1$: $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(-1)^n}{n} = -\sum_{n=1}^{\infty}\frac{1}{n}$ diverges

The interval of convergence is $-1 < x \le 1$.

(c) $1 + x = \frac{3}{2}$, so $x = \frac{1}{2}$.

Because this is an alternating series with terms decreasing to 0 in absolute value, the error satisfies

$$|E_5| \le \left|\frac{(\frac{1}{2})^6}{6}\right| = \frac{1}{384} \approx 0.0026.$$

Or using a Lagrange error bound, there is a c with $0 \le c \le \frac{1}{2}$ such that

$$|E_5| = \left|\frac{f^{(6)}(c)(\frac{1}{2})^6}{6!}\right| = \frac{5!(1+c)^{-6}}{6!2^6} = \frac{1}{(1+c)^6(6)2^6} \le \frac{1}{6\cdot 2^6}$$

(d) $\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x^2)^n}{n} = \frac{1}{2}\ln(1+x^2) = \ln\sqrt{1+x^2}$

1982 BC6

Point $P(x, y)$ moves in the xy-plane in such a way that $\frac{dx}{dt} = \frac{1}{t+1}$ and $\frac{dy}{dt} = 2t$ for $t \geq 0$.

(a) Find the coordinates of P in terms of t if, when $t = 1$, $x = \ln 2$ and $y = 0$.

(b) Write an equation expressing y in terms of x.

(c) Find the average rate of change of y with respect to x as t varies from 0 to 4.

(d) Find the instantaneous rate of change of y with respect to x when $t = 1$.

1982 BC6
Solution

(a) $x = \int \frac{1}{t+1}\,dt = \ln(t+1) + C_1$

$\ln 2 = x(1) = \ln 2 + C_1, \text{ so } C_1 = 0$

$x = \ln(t+1)$

$y = \int 2t\,dt = t^2 + C_2$

$0 = y(1) = 1 + C_2, \text{ so } C_2 = -1$

$y = t^2 - 1$

(b) $e^x = t + 1, \text{ so } y = (e^x - 1)^2 - 1 = e^{2x} - 2e^x$

(c) $\frac{y(4) - y(0)}{x(4) - x(0)} = \frac{15 - (-1)}{\ln 5 - \ln 1} = \frac{16}{\ln 5}$

(d) At $t = 1$, $\frac{dy}{dx} = \frac{2t}{\frac{1}{t+1}} = 4$

or

From (b), using $x = \ln 2$ when $t = 1$

$\frac{dy}{dx} = 2e^{2\ln 2} - 2e^{\ln 2} = 8 - 4 = 4$

1982 BC7

Let f be the function given by $f(x)=\begin{cases} x^2\sin\left(\dfrac{1}{x}\right), & \text{for } x\neq 0 \\ 0, & \text{for } x=0 \end{cases}$

(a) Using the definition of the derivative, prove that f is differentiable at $x=0$.

(b) Find $f'(x)$ for $x\neq 0$.

(c) Show that f' is not continuous at $x=0$.

1982 BC7
Solution

(a) $f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x} = \lim_{x\to 0}\frac{x^2\sin\left(\frac{1}{x}\right)-0}{x} = \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$

since $\sin\left(\frac{1}{x}\right)$ is bounded implies that $\left|x\sin\left(\frac{1}{x}\right)\right| \le |x|$ for all $x \ne 0$.

(b) For $x \ne 0$,

$f'(x) = 2x\sin\left(\frac{1}{x}\right) + x^2\left[\cos\left(\frac{1}{x}\right)\right]\left(-x^{-2}\right) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$

(c) $\lim_{x\to 0} f'(x) = \lim_{x\to 0}\left(2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right)$

This limit <u>does not exist</u> since $\cos\left(\frac{1}{x}\right)$ oscillates between 1 and -1 as $x \to 0$.

Therefore f' is not continuous at $x = 0$.

1983 AB1

Let *f* be the function defined by $f(x) = -2 + \ln(x^2)$.

(a) For what real numbers x is *f* defined?

(b) Find the zeros of *f*.

(c) Write an equation for the line tangent to the graph of *f* at $x = 1$.

1983 AB1
Solution

(a) $\ln u$ is defined only for $u > 0$.
$x^2 > 0$ except for $x = 0$.
Therefore $f(x)$ is defined for all $x \neq 0$.

(b) $f(x) = 0$ when $\ln(x^2) = 2$.
$x^2 = e^2$
$|x| = e$
The zeros are $x = \pm e$.

(c) $f'(x) = \frac{2x}{x^2} = \frac{2}{x}$
$f'(1) = \frac{2}{1} = 2$
$f(1) = -2 + \ln(1^2) = -2$

The equation of the tangent line is
$y - (-2) = 2(x - 1)$ or $y = 2x - 4$

1983 AB2

A particle moves along the x-axis so that at time t its position is given by $x(t) = t^3 - 6t^2 + 9t + 11$.

(a) What is the velocity of the particle at $t = 0$?

(b) During what time intervals is the particle moving to the left?

(c) What is the total distance traveled by the particle from $t = 0$ to $t = 2$?

1983 AB2
Solution

(a) $v(t) = x'(t) = 3t^2 - 12t + 9$

$v(0) = 9$

(b) The particle is moving to the left when $v(t) < 0$.

$$3t^2 - 12t + 9 < 0$$
$$t^2 - 4t + 3 < 0$$
$$(t-1)(t-3) < 0$$

The particle is moving to the left on the interval $1 < t < 3$.

(c) Distance = $(x(1) - x(0)) + (x(1) - x(2)) = 15 - 11 + 15 - 13 = 6$

or

$$\begin{aligned}\text{Distance} &= \int_0^2 |v(t)|\,dt \\ &= \int_0^1 v(t)\,dt - \int_1^2 v(t)\,dt \\ &= \left(t^3 - 6t^2 + 9t\Big|_0^1\right) - \left(t^3 - 6t^2 + 9t\Big|_1^2\right) \\ &= 4 - (-2) = 6\end{aligned}$$

1983 AB3/BC1

Let f be the function defined for $\frac{\pi}{6} \le x \le \frac{5\pi}{6}$ by $f(x) = x + \sin^2 x$.

(a) Find all values of x for which $f'(x) = 1$.

(b) Find the x-coordinates of all minimum points of f. Justify your answer.

(c) Find the x-coordinates of all inflection points of f. Justify your answer.

1983 AB3/BC1
Solution

(a) $f'(x) = 1 + 2\sin x \cos x = 1 + \sin 2x$

$1 = 1 + \sin 2x$

$x = \frac{\pi}{2}$ is the only solution in the interval $\frac{\pi}{6} \le x \le \frac{5\pi}{6}$.

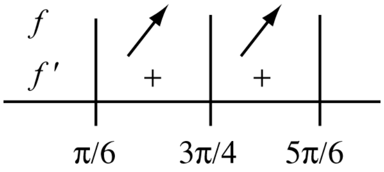
(b) $f'(x) = 1 + \sin 2x = 0$, so $x = \frac{3\pi}{4}$

The minimum occurs at the critical point or at the endpoints.

critical point: $f\left(\frac{3\pi}{4}\right) = \frac{3\pi}{4} + \frac{1}{2} = 2.856$

endpoints: $f\left(\frac{\pi}{6}\right) = \frac{\pi}{6} + \frac{1}{4} = 0.774$

$f\left(\frac{5\pi}{6}\right) = \frac{5\pi}{6} + \frac{1}{4} = 2.868$



Therefore the minimum is at $x = \frac{\pi}{6}$.

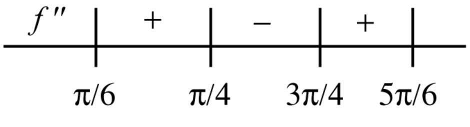
or

Since $f'(x) = 1 + \sin 2x \ge 0$ for all *x*, the function *f* is increasing on the entire interval. Therefore the minimum is at $x = \frac{\pi}{6}$.

(c) $f''(x) = 2\cos 2x$

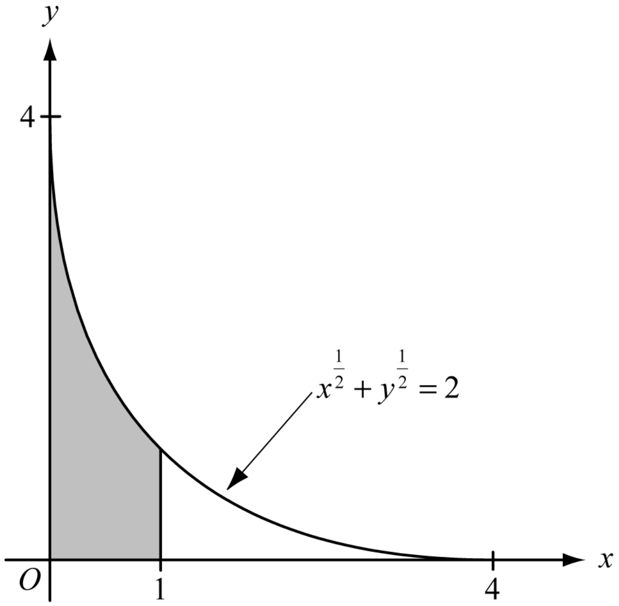
$2\cos 2x = 0$

$x = \frac{\pi}{4}, \frac{3\pi}{4}$



Therefore the inflection points occur at $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$ since this is where f'' changes sign from positive to negative and from negative to positive, respectively.

1983 AB4



The figure above shows the graph of the equation $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 2$. Let R be the shaded region between the graph of $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 2$ and the x-axis from $x = 0$ to $x = 1$.

(a) Find the area of R by setting up and integrating a definite integral.

(b) Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid formed by revolving the region R about the x-axis.

(c) Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid formed by revolving the region R about the line $x = 1$.

1983 AB4
Solution

(a) $y = \left(2 - x^{1/2}\right)^2 = 4 - 4x^{1/2} + x$

$$\text{Area } = \int_0^1 \left(2 - x^{1/2}\right)^2 dx = \int_0^1 \left(4 - 4x^{1/2} + x\right) dx = \left(4x - \frac{8}{3}x^{3/2} + \frac{x^2}{2}\right)\Bigg|_0^1 = \frac{11}{6}$$

or

$$\text{Area} = = \int_0^1 1\,dy + \int_1^4 \left(4 - 4y^{1/2} + y\right) dy = 1 + \left(4y - \frac{8}{3}y^{3/2} + \frac{1}{2}y^2\right)\Bigg|_1^4 = \frac{11}{6}$$

(b) $$\text{Volume } = \pi\int_0^1 \left(2 - x^{1/2}\right)^4 dx$$

$$= \pi\int_0^1 \left(16 - 32x^{1/2} + 24x - 8x^{3/2} + x^2\right) dx$$

or

$$\text{Volume } = 2\pi\int_0^1 y\,dy + 2\pi\int_1^4 y\left(2 - y^{1/2}\right)^2 dy$$

$$= \pi + \int_1^4 \left(4y - 4y^{3/2} + y^2\right) dy$$

(c) $$\text{Volume } = 2\pi\int_0^1 (2 - x^{1/2})^2 (1 - x)\,dx$$

$$= 2\pi\int_0^1 \left(4 - 4x^{1/2} - 3x + 4x^{3/2} - x^2\right) dx$$

or

$$\text{Volume } = \pi\int_0^1 1\,dy + \pi\int_1^4 \left(1 - (1 - x)^2\right) dy$$

$$= \pi + \pi\int_1^4 \left(2x - x^2\right) dy$$

$$= \pi + \pi\int_1^4 \left(2(2 - y^{1/2})^2 - (2 - y^{1/2})^4\right) dy$$

$$= \pi + \pi\int_1^4 \left(-8 + 24y^{1/2} - 22y + 8y^{3/2} - y^2\right) dy$$

1983 AB5/BC3

At time $t = 0$, a jogger is running at a velocity of 300 meters per minute. The jogger is slowing down with a negative acceleration that is directly proportional to time t. This brings the jogger to a stop in 10 minutes.

(a) Write an expression for the velocity of the jogger at time t.

(b) What is the total distance traveled by the jogger in that 10-minute interval?

1983 AB5/BC3
Solution

(a) $a = -kt$

$v = -\frac{kt^2}{2} + C$

$300 = v(0) = C$

$0 = v(10) = -\frac{100}{2}k + 300$

$k = 6$

Therefore $v(t) = -3t^2 + 300$

(b) $s(t) = -t^3 + 300t$

Distance = $s(10) - s(0) = 2000$ meters

1983 BC2

Consider the curve $y = 2x^{\frac{1}{2}}$ from $x = 3$ to $x = 8$.

(a) Set up, but do not integrate, an integral expression in terms of a single variable for the length of the curve.

(b) Let S be the surface generated by revolving the curve about the x-axis. Find the area of S by setting up and integrating a definite integral.

1983 BC2
Solution

(a) $y' = x^{-1/2}$

Length = $\int_3^8 \sqrt{1+x^{-1}}\,dx = \int_3^8 \sqrt{\dfrac{x+1}{x}}\,dx$

(b) Surface area = $2\pi\int_3^8 \left(2x^{1/2}\right)\sqrt{\dfrac{x+1}{x}}dx$

$$= 4\pi\int_3^8 (x+1)^{1/2}\,dx$$

$$= (4\pi)\left(\frac{2}{3}\right)(x+1)^{3/2}\Big|_3^8 = \frac{8\pi}{3}(27-8) = \frac{152\pi}{3}$$

1983 BC4

Consider the differential equation $\frac{dy}{dx} + 2xy = xe^{\left(-x^2+x\right)}$.

(a) Find the general solution of the differential equation.

(b) Find the particular solution of the differential equation that satisfies the condition $y = 3$ when $x = 0$.

1983 BC4
Solution

(a) $\dfrac{dy}{dx}+2xy=xe^{-x^2+x}$

The integrating factor is $e^{\int 2x\,dx}=e^{x^2}$

$e^{x^2}\dfrac{dy}{dx}+2xe^{x^2}y=xe^{x}$

$\dfrac{d(ye^{x^2})}{dx}=xe^{x}$

$ye^{x^2}=\int xe^{x}\,dx=xe^{x}-e^{x}+C$

$y=xe^{-x^2+x}-e^{-x^2+x}+Ce^{-x^2}$

or

$y=\dfrac{xe^{x}-e^{x}+C}{e^{x^2}}$

(b) $3=-1+C$

$C=4$

$y=xe^{-x^2+x}-e^{-x^2+x}+4e^{-x^2}$

1983 BC5

Consider the power series $\sum_{n=0}^{\infty} a_n x^n$, where $a_0 = 1$ and $a_n = \left(\frac{7}{n}\right) a_{n-1}$ for $n \geq 1$.

(a) Find the first four terms and the general term of the series.

(b) For what values of x does the series converge?

(c) If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, find the value of $f'(1)$.

1983 BC5
Solution

(a) $1+7x+\dfrac{7^2x^2}{2!}+\dfrac{7^3x^3}{3!}+\cdots+\dfrac{7^nx^n}{n!}+\cdots$

(b) $\lim\limits_{n\to\infty}\left|\dfrac{7^{n+1}x^{n+1}}{(n+1)!}\cdot\dfrac{n!}{7^nx^n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{7x}{n+1}\right| = 0 < 1$

Therefore the series converges for all real x.

or

$e^x = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$ converges for all real x.

Therefore $e^{7x} = \displaystyle\sum_{n=0}^{\infty}\frac{(7x)^n}{n!}$ converges for all real x.

(c)
$$f'(x) = 7+7^2x+\frac{7^3x^2}{2!}+\frac{7^4x^3}{3!}+\cdots+\frac{7^nx^{n-1}}{(n-1)!}+\cdots$$
$$f'(1) = 7+7^2+\frac{7^3}{2!}+\frac{7^4}{3!}+\cdots+\frac{7^n}{(n-1)!}+\cdots$$
$$= 7\left(1+7^2+\frac{7^2}{2!}+\frac{7^3}{3!}+\ldots+\frac{7^{n-1}}{(n-1)!}+\ldots\right)$$
$$= 7e^7$$

1984 AB1

A particle moves along the x-axis so that, at any time $t \geq 0$, its acceleration is given by $a(t) = 6t + 6$. At time $t = 0$, the velocity of the particle is –9, and its position is –27.

(a) Find $v(t)$, the velocity of the particle at any time $t \geq 0$.

(b) For what values of $t \geq 0$ is the particle moving to the right?

(c) Find $x(t)$, the position of the particle at any time $t \geq 0$.

1984 AB1
Solution

(a) $a(t) = 6t + 6$

$v(t) = 3t^2 + 6t + C$

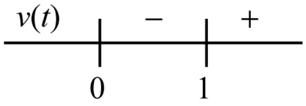
$v(0) = -9 \Rightarrow C = -9$

$v(t) = 3t^2 + 6t - 9$

(b) $3t^2 + 6t - 9 > 0$

$t^2 + 2t - 3 > 0$

$(t+3)(t-1) > 0$



The particle is moving to the right for $t > 1$.

(c) $x(t) = t^3 + 3t^2 - 9t + C_1$

$-27 = x(0) = C_1$

$x(t) = t^3 + 3t^2 - 9t - 27$

1984 AB2

Let f be the function defined by $f(x) = \dfrac{x + \sin x}{\cos x}$ for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$.

(a) State whether f is an even or an odd function. Justify your answer.

(b) Find $f'(x)$.

(c) Write an equation of the line tangent to the graph of f at the point where $x = 0$.

1984 AB2
Solution

(a) f is an <u>odd</u> function because

$$f(-x)=\frac{-x+\sin(-x)}{\cos(-x)}=\frac{-x-\sin x}{\cos x}=-f(x)$$

(b) $$f'(x)=\frac{\cos x(1+\cos x)-(x+\sin x)(-\sin x)}{\cos^2 x}$$
$$=\frac{1+\cos x+x\sin x}{\cos^2 x}$$

(c) $$f'(0)=\frac{1+\cos 0+0\cdot\sin 0}{\cos^2 0}=2$$
$$f(0)=\frac{0+\sin 0}{\cos 0}=0$$

The equation of the tangent line is $y-0=2(x-0)$ or $y=2x$.

1984 AB3/BC1

Let R be the region enclosed by the x-axis, the y-axis, the line $x = 2$, and the curve $y = 2e^{x} + 3x$.

(a) Find the area of R by setting up and evaluating a definite integral. Your work must include an antiderivative.

(b) Find the volume of the solid generated by revolving R about the $\underline{y\text{-axis}}$ by setting up and evaluating a definite integral. Your work must include an antiderivative.

1984 AB3/BC1
Solution

(a) Area $= \int_0^2 (2e^x + 3x)\,dx$

$$= (2e^x + \frac{3}{2}x^2)\Big|_0^2$$

$$= 2e^2 + 6 - 2e^0$$

$$= 2e^2 + 4$$

(b) Volume $= 2\pi \int_0^2 x(2e^x + 3x)\,dx$

$$= 2\pi \int_0^2 (2xe^x + 3x^2)\,dx$$

$$= 2\pi\left(2xe^x\Big|_0^2 - 2\int_0^2 e^x dx + \int_0^2 3x^2 dx \right)$$

$$= 2\pi\left(2(x-1)e^x + x^3 \right)\Big|_0^2$$

$$= 2\pi\left((2e^2 + 8) - 2(-e^0) \right)$$

$$= 4\pi(e^2 + 5)$$

1984 AB4/BC3

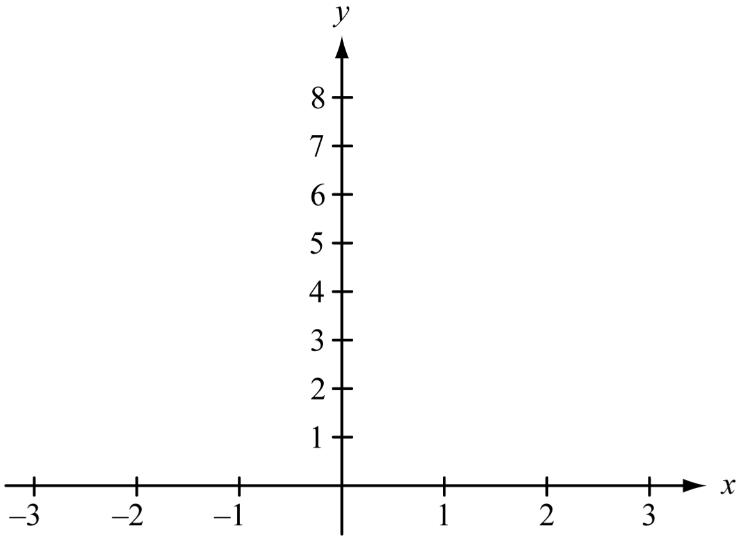
A function f is continuous on the closed interval $[-3,\ 3]$ such that $f(-3)=4$ and $f(3)=1$. The functions f' and f'' have the properties given in the table below.

x	$-3<x<-1$	$x=-1$	$-1<x<1$	$x=1$	$1<x<3$
$f'(x)$	Positive	Fails to exist	Negative	0	Negative
$f''(x)$	Positive	Fails to exist	Positive	0	Negative

(a) What are the x-coordinates of all absolute maximum and absolute minimum points of f on the interval $[-3,\ 3]$? Justify your answer.

(b) What are the x-coordinates of all points of inflection of f on the interval $[-3,\ 3]$? Justify your answer.

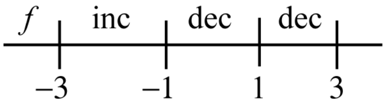
(c) On the axes provided, sketch a graph that satisfies the given properties of f.



1984 AB4/BC3
Solution

(a) The absolute maximum occurs at $x = -1$ because f is increasing on the interval $[-3, -1]$ and decreasing on the interval $[-1, 3]$.

or



The absolute minimum must occur at $x = 1$ (the other critical point) or at an endpoint. However, f is decreasing on the interval $[-1, 3]$. Therefore the absolute minimum is at an endpoint. Since $f(-3) = 4 > 1 = f(3)$, the absolute minimum is at $x = 3$.

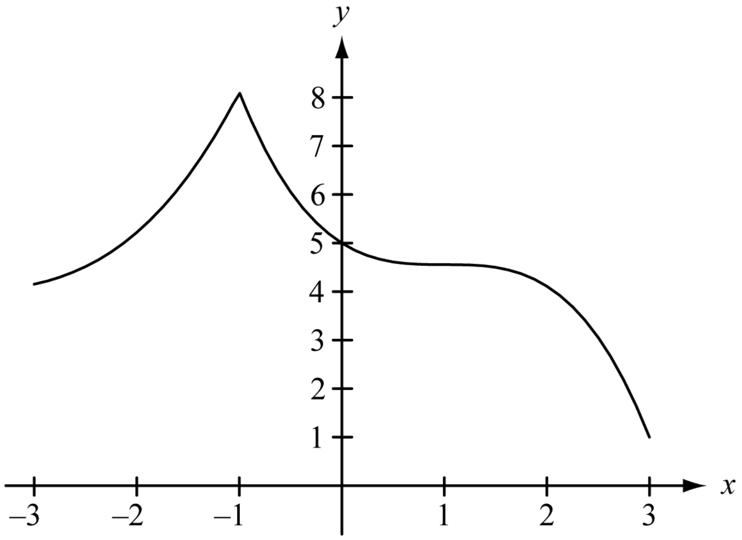
(b) There is an inflection point at $x = 1$ because:

the graph of f changes from concave up to concave down at $x = 1$

or

f'' changes sign from positive to negative at $x = 1$

(c) This is one possibility:



1984 AB5

The volume V of a cone $\left(V = \frac{1}{3}\pi r^2 h\right)$ is increasing at the rate of 28π cubic units per second. At the instant when the radius r of the cone is 3 units, its volume is 12π cubic units and the radius is increasing at $\frac{1}{2}$ unit per second.

(a) At the instant when the radius of the cone is 3 units, what is the rate of change of the area of its base?

(b) At the instant when the radius of the cone is 3 units, what is the rate of change of its height h?

(c) At the instant when the radius of the cone is 3 units, what is the instantaneous rate of change of the area of its base with respect to its height h?

1984 AB5
Solution

(a) $A = \pi r^2$

When $r = 3$, $\dfrac{dA}{dt} = 2\pi r \dfrac{dr}{dt} = 2\pi \cdot 3 \cdot \dfrac{1}{2} = 3\pi$

(b) $V = \dfrac{1}{3}\pi r^2 h$

$$\frac{dV}{dt} = \frac{1}{3}\pi r^2 \frac{dh}{dt} + \frac{2}{3}\pi r h \frac{dr}{dt}$$

$$28\pi = \frac{1}{3}\pi(9)\frac{dh}{dt} + \frac{2}{3}\pi(3)(4)\left(\frac{1}{2}\right)$$

$$\frac{dh}{dt} = 8$$

or

$V = \dfrac{1}{3}Ah$

$$\frac{dV}{dt} = \frac{1}{3}A\frac{dh}{dt} + \frac{1}{3}h\frac{dA}{dt}$$

$$28\pi = \frac{1}{3}(9\pi)\frac{dh}{dt} + \frac{1}{3}4(3\pi)$$

$$\frac{dh}{dt} = 8$$

(c) $\dfrac{dA}{dh} = \dfrac{\;\dfrac{dA}{dt}\;}{\dfrac{dh}{dt}} = \dfrac{3\pi}{8}$

or

$A = \pi r^2$

$$\frac{dA}{dh} = 2\pi r \frac{dr}{dh}$$

$$\frac{dr}{dh} = \frac{\;\dfrac{dr}{dt}\;}{\dfrac{dh}{dt}} = \frac{1/2}{8} = \frac{1}{16}$$

Therefore $\dfrac{dA}{dh} = 2\pi(3)\left(\dfrac{1}{16}\right) = \dfrac{3\pi}{8}$

1984 BC2

The path of a particle is given for time $t > 0$ by the parametric equations $x = t + \frac{2}{t}$ and $y = 3t^2$.

(a) Find the coordinates of each point on the path where the velocity of the particle in the x direction is zero.

(b) Find $\frac{dy}{dx}$ when $t = 1$.

(c) Find $\frac{d^2y}{dx^2}$ when $y = 12$.

1984 BC2
Solution

(a) $\frac{dx}{dt} = 1 - \frac{2}{t^2}$

$\frac{dx}{dt} = 0$ when $t = \sqrt{2}$. The corresponding point on the path is $(2\sqrt{2}, 6)$.

(b) $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t}{1 - \frac{2}{t^2}}$

At $t = 1$, $\frac{dy}{dx} = -6$

(c) $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{6\left(1 - \frac{2}{t^2}\right) - 6t\left(\frac{4}{t^3}\right)}{\left(1 - \frac{2}{t^2}\right)^3}$

When $y = 12$, $t = 2$ and so $\frac{d^2y}{dx^2} = \frac{6\left(\frac{1}{2}\right) - 6}{\left(\frac{1}{2}\right)^3} = -24$.

1984 BC4

Let f be the function defined by $f(x)=\sum_{n=1}^{\infty}\frac{x^n n^n}{3^n n!}$ for all values of x for which the series converges.

(a) Find the radius of convergence of this series.

(b) Use the first three terms of this series to find an approximation of $f(-1)$.

(c) Estimate the amount of error involved in the approximation in part (b). Justify your answer.

1984 BC4
Solution

(a) $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{x^{n+1}(n+1)^{n+1}}{3^{n+1}(n+1)!}}{\dfrac{x^n n^n}{3^n n!}}\right| = \lim_{n\to\infty}\left|\frac{x}{3}\left(\frac{n+1}{n}\right)^n\right| = \left|\frac{x}{3}\cdot e\right| < 1$$

Since the series converges for $|x| < \frac{3}{e}$, the radius of convergence is $\frac{3}{e}$.

(b) $$f(-1) = \sum_{n=1}^{\infty}\frac{(-1)^n n^n}{3^n n!} = -\frac{1}{3}+\frac{2}{9}-\frac{1}{6}+\ldots$$
$$\approx -\frac{5}{18}$$

(c) The series is alternating with the absolute value of the terms decreasing to 0. Therefore the error is less than the absolute value of the first omitted term. Hence

$$\left|f(-1)-\left(-\frac{5}{18}\right)\right| < \frac{4^4}{3^4 4!} = \frac{32}{243}$$

1984 BC5

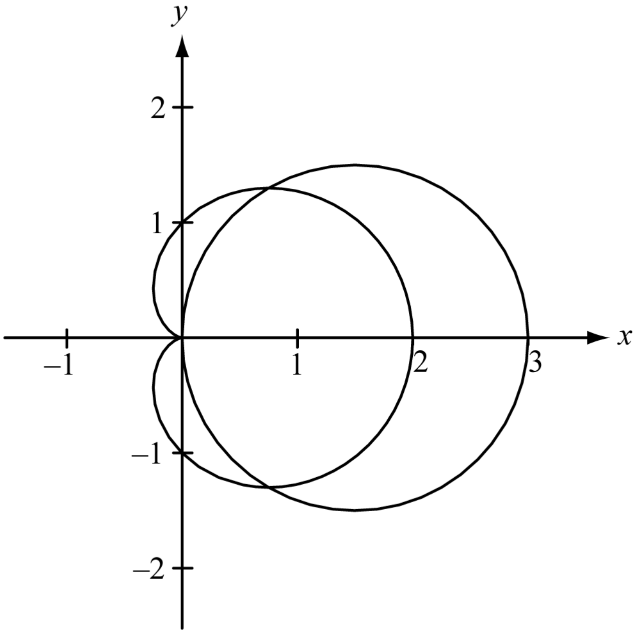
Consider the curves $r = 3\cos\theta$ and $r = 1 + \cos\theta$.

(a) Sketch the curves on the same set of axes.

(b) Find the area of the region inside the curve $r = 3\cos\theta$ and outside the curve $r = 1 + \cos\theta$ by setting up and evaluating a definite integral. Your work must include an antiderivative.

1984 BC5
Solution

(a)



(b) The intersection occurs when $3\cos\theta = 1 + \cos\theta$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \pm\frac{\pi}{3}$$

$$\begin{aligned}
\text{Area } &= \frac{1}{2}\int_{-\pi/3}^{\pi/3}\left((3\cos\theta)^2 - (1+\cos\theta)^2\right)d\theta \\
&= \frac{1}{2}\int_{-\pi/3}^{\pi/3}(8\cos^2\theta - 1 - 2\cos\theta)\,d\theta \\
&= \frac{1}{2}\int_{-\pi/3}^{\pi/3}(4 + 4\cos 2\theta - 1 - 2\cos\theta)\,d\theta \\
&= \frac{1}{2}\int_{-\pi/3}^{\pi/3}(3 + 4\cos 2\theta - 2\cos\theta)\,d\theta \\
&= \frac{1}{2}(3\theta + 2\sin 2\theta - 2\sin\theta)\Big|_{-\pi/3}^{\pi/3} \\
&= \frac{1}{2}\left(\pi + 2\sin\frac{2\pi}{3} - 2\sin\frac{\pi}{3} - (-\pi - 2\sin\frac{2\pi}{3} + \sin\frac{\pi}{3})\right) \\
&= \pi
\end{aligned}$$

1985 AB1

Let f be the function defined by $f(x) = \dfrac{2x-5}{x^2-4}$.

(a) Find the domain of f.

(b) Write an equation for each vertical and each horizontal asymptote for the graph of f.

(c) Find $f'(x)$.

(d) Write an equation for the line tangent to the graph of f at the point $(0, f(0))$.

1985 AB1
Solution

(a) The domain of f is all real numbers except $x = 2$ and $x = -2$.

(b) Asymptotes

Vertical: $x = 2,\ x = -2$

Horizontal: $y = 0$

(c) $$f'(x) = \frac{2(x^2-4) - 2x(2x-5)}{(x^2-4)^2} = \frac{-2x^2+10x-8}{(x^2-4)^2} = \frac{-2(x-4)(x-1)}{(x^2-4)^2}$$

(d) Tangent line at $x = 0$

$$f(0) = \frac{5}{4}$$

$$f'(0) = -\frac{1}{2}$$

The equation of the line is

$$y - \frac{5}{4} = -\frac{1}{2}(x-0)$$

or

$$y = -\frac{1}{2}x + \frac{5}{4}$$

or

$$2x + 4y = 5$$

1985 AB2/BC1

A particle moves along the x-axis with acceleration given by $a(t) = \cos t$ for $t \geq 0$. At $t = 0$, the velocity $v(t)$ of the particle is 2, and the position $x(t)$ is 5.

(a) Write an expression for the velocity $v(t)$ of the particle.

(b) Write an expression for the position $x(t)$.

(c) For what values of t is the particle moving to the right? Justify your answer.

(d) Find the total distance traveled by the particle from $t = 0$ to $t = \frac{\pi}{2}$.

1985 AB2/BC1
Solution

(a) $v(t) = \sin(t) + C$

$2 = \sin(0) + C$

$C = 2$

$v(t) = \sin(t) + 2$

(b) $x(t) = -\cos(t) + 2t + C$

$5 = -\cos(0) + 2(0) + C$

$C = 6$

$x(t) = -\cos(t) + 2t + 6$

(c) The particle moves to the right when $v(t) > 0$,. i.e. when $\sin(t) + 2 > 0$.
This is true for all $t \geq 0$ because

$-1 \leq \sin(t) \leq 1 \Rightarrow 0 < -1 + 2 \leq \sin(t) + 2 \leq 1 + 2$ for all t.

(d) The particle never changes directions since it moves to the right for all $t \geq 0$.

$x(0) = -\cos(0) + 2(0) + 6 = 5$

$x\left(\frac{\pi}{2}\right) = -\cos\left(\frac{\pi}{2}\right) + 2\left(\frac{\pi}{2}\right) + 6 = \pi + 6$

Distance = $x\left(\frac{\pi}{2}\right) - x(0) = \pi + 1$

or

Distance = $\int_0^{\pi/2} |v(t)|\, dt = \int_0^{\pi/2} |\sin(t) + 2|\, dt$

$= \int_0^{\pi/2} (\sin(t) + 2)\, dt = (-\cos t + 2t)\Big|_0^{\pi/2} = \pi + 1$

1985 AB3

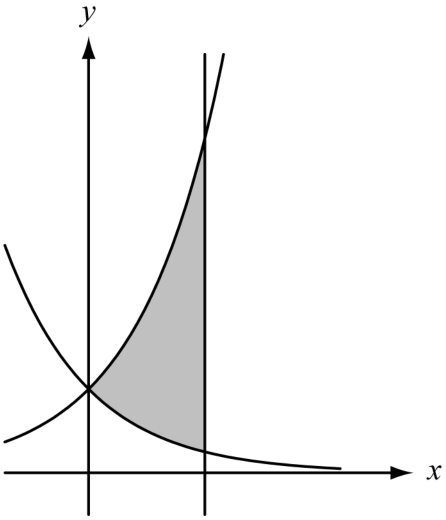
Let R be the region enclosed by the graphs of $y = e^{-x}$, $y = e^{x}$, and $x = \ln 4$.

(a) Find the area of R by setting up and evaluating a definite integral.

(b) Set up, but do not integrate, an integral expression in terms of a single variable for the volume generated when the region R is revolved about the x-axis.

(c) Set up, but do not integrate, an integral expression in terms of a single variable for the volume generated when the region R is revolved about the y-axis.

1985 AB3
Solution



(a) Intersection is when $x = 0$.

$$\text{Area} = \int_0^{\ln 4}\left(e^x - e^{-x}\right)dx$$
$$= \left(e^x + e^{-x}\right)\Big|_0^{\ln 4}$$
$$= \left(4 + \frac{1}{4}\right) - (1+1) = \frac{9}{4}$$

(b) Disks:

$$\text{Volume} = \pi\int_0^{\ln 4}\left(e^{2x} - e^{-2x}\right)dx$$

or

Shells:

$$\text{Volume} = 2\pi\int_{1/4}^{1} y(\ln 4 + \ln y)\,dy + 2\pi\int_1^4 y(\ln 4 - \ln y)\,dy$$

(c) Disks:

$$\text{Volume} = \pi\int_{1/4}^{1}\left((\ln 4)^2 - (-\ln y)^2\right)dy + \pi\int_1^4\left((\ln 4)^2 - (\ln y)^2\right)dy$$
$$= \pi\int_{1/4}^{4}(\ln 4)^2 dy - \pi\int_{1/4}^{4}(\ln y)^2\,dy$$
$$= \frac{15}{4}\pi(\ln 4)^2 - \pi\int_{1/4}^{4}(\ln y)^2\,dy$$

or

Shells:

$$\text{Volume} = 2\pi\int_0^{\ln 4} x\left(e^x - e^{-x}\right)dx$$

1985 AB4/BC3

Let $f(x) = 14\pi x^2$ and $g(x) = k^2 \sin\left(\frac{\pi x}{2k}\right)$ for $k > 0$.

(a) Find the average value of f on $\left[1, 4\right]$.

(b) For what value of k will the average value of g on $\left[0,\, k\right]$ be equal to the average value of f on $\left[1, 4\right]$?

1985 AB4/BC3
Solution

(a) Average value $= \frac{1}{3}\int_1^4 14\pi x^2\,dx$

$$= \frac{14\pi}{3}\frac{x^3}{3}\Big|_1^4 = \frac{14\pi}{9}(64-1)$$

$$= 98\pi$$

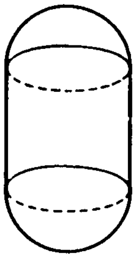
(b) Average value $= \frac{1}{k}\int_0^k k^2 \sin\frac{\pi x}{2k}\,dx$

$$= -\frac{2k^2}{\pi}\cos\frac{\pi x}{2k}\Big|_0^k = -\frac{2k^2}{\pi}(0-1) = \frac{2k^2}{\pi}$$

Therefore $\frac{2k^2}{\pi} = 98\pi$.

Hence $k^2 = 49\pi^2$ and so $k = 7\pi$.

1985 AB5/BC2



The balloon shown is in the shape of a cylinder with hemispherical ends of the same radius as that of the cylinder. The balloon is being inflated at the rate of 261π cubic centimeters per minute. At the instant the radius of the cylinder is 3 centimeters., the volume of the balloon is 144π cubic centimeters and the radius of the cylinder is increasing at the rate of 2 centimeters per minute. (The volume of a cylinder is $\pi r^2 h$ and the volume of a sphere is $\frac{4}{3}\pi r^3$).

(a) At this instant, what is the height of the cylinder?

(b) At this instant, how fast is the height of the cylinder increasing?

1985 AB5/BC2
Solution

(a) $V = \pi r^2 h + \frac{4}{3}\pi r^3$

$144\pi = \pi(3)^2 h + \frac{4}{3}\pi(3)^3$

$h = 12$

At this instant, the height is 12 centimeters.

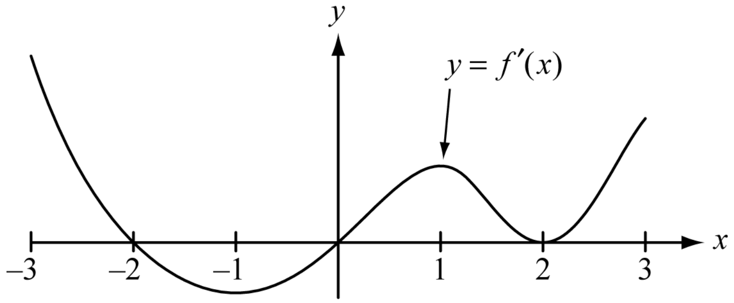
(b) $\frac{dV}{dt} = \pi r^2 \frac{dh}{dt} + 2\pi r h \frac{dr}{dt} + 4\pi r^2 \frac{dr}{dt}$

$261\pi = \pi(3)^2 \frac{dh}{dt} + 2\pi(3)(12)(2) + 4\pi(3)^2(2)$

$\frac{dh}{dt} = 5$

At this instant, the height is increasing at the rate of 5 centimeters per minute.

1985 AB6



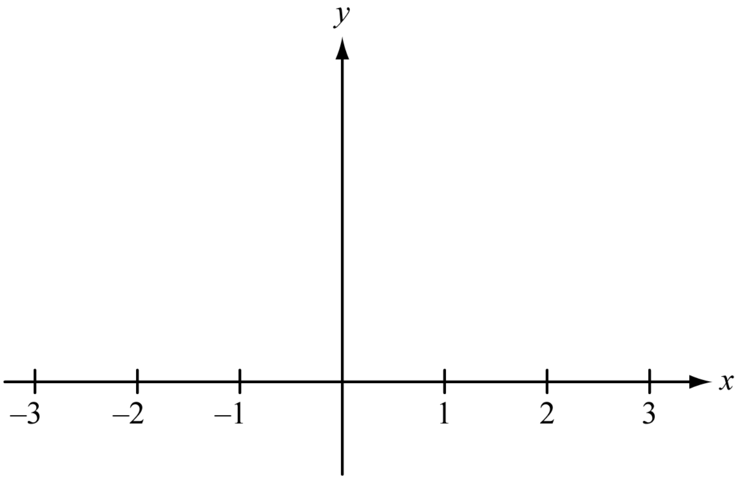
Note: This is the graph of the derivative of f, not the graph of f.

The figure above shows the graph of f', the derivative of a function f. The domain of the function f is the set of all x such that $-3 \le x \le 3$.

(a) For what values of x, $-3 < x < 3$, does f have a relative maximum? A relative minimum? Justify your answer.

(b) For what values of x is the graph of f concave up? Justify your answer.

(c) Use the information found in parts (a) and (b) and the fact that $f(-3) = 0$ to sketch a possible graph of f on the axes provided below.



1985 AB6
Solution

(a) f has a relative maximum at $x = -2$ because:

f' changes from positive to negative at $x = -2$

or

f changes from increasing to decreasing at $x = -2$

or

$f'(-2) = 0$ and $f''(-2) < 0$

f has a relative minimum at $x = 0$ because:

f' changes from negative to positive at $x = 0$.

or

f changes from decreasing to increasing at $x = 0$.

or

$f'(0) = 0$ and $f''(0) > 0$

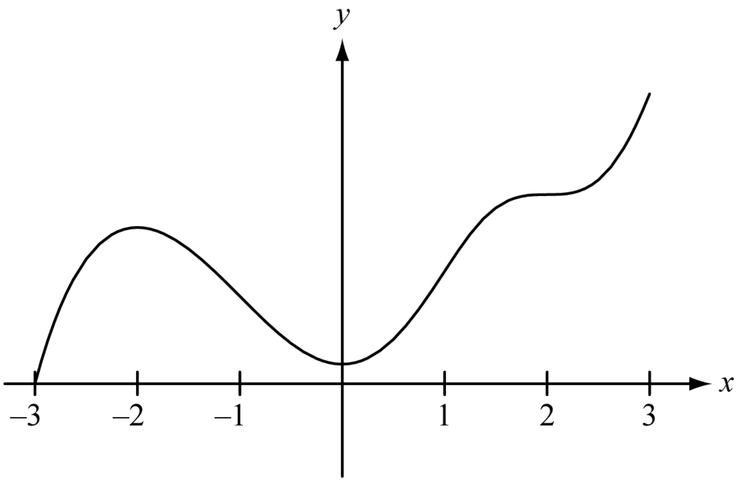
(b) f is concave up on $(-1,1)$ and $(2,3)$ because:

f' is increasing on those intervals

or

$f'' > 0$ on those intervals

(c)



1985 BC4

Given the differential equation $\frac{dy}{dx} = \frac{-xy}{\ln y}$, $y > 0$.

(a) Find the general solution of the differential equation.

(b) Find the solution that satisfies the condition that $y = e^2$ when $x = 0$. Express your answer in the form $y = f(x)$.

(c) Explain why $x = 2$ is not in the domain of the solution found in part (b).

1985 BC4
Solution

(a) $\frac{\ln y}{y} dy = -x\, dx$

$\frac{(\ln y)^2}{2} = -\frac{x^2}{2} + C$

or

$(\ln y)^2 = -x^2 + C$

or

$\ln y = \pm\sqrt{C - x^2}$

or

$y = e^{\pm\sqrt{C-x^2}}$

(b) $(\ln e^2)^2 = 0 + C$

$C = 4$

$(\ln y)^2 = 4 - x^2$

$\ln y = \pm\sqrt{4 - x^2}$

But $x = 0, y = e^2 \Rightarrow \ln y = \sqrt{4 - x^2}$

$y = e^{\sqrt{4-x^2}}$

(c) If $x = 2$, then $y = 1$ and $\ln y = 0$. This causes $\frac{-xy}{\ln y}$ to be undefined.

1985 BC5

Let f be the function defined by $f(x) = -\ln x$ for $0 < x \leq 1$ and let R be the region between the graph of f and the x-axis.

(a) Determine whether region R has finite area. Justify your answer.

(b) Determine whether the solid generated by revolving region R about the y-axis has finite volume. Justify your answer.

1985 BC5
Solution

(a)
$$\begin{aligned}\int_0^1 -\ln x\,dx &= -\lim_{a\to 0^+}\int_a^1 \ln x\,dx\\ &= \lim_{a\to 0^+}\left[x - x\ln x\right]_a^1\\ &= \lim_{a\to 0^+}\left[1 - a + a\ln a\right]\\ &= 1 + \lim_{a\to 0^+}\frac{\frac{1}{a}}{-\frac{1}{a^2}} = 1\end{aligned}$$

The area is finite.

(b)
$$\begin{aligned}2\pi\int_0^1 x\left(-\ln x\right)dx &= 2\pi\lim_{a\to 0^+}\int_a^1 x(-\ln x)\,dx\\ &= 2\pi\lim_{a\to 0^+}\left[\frac{-x^2}{2}\ln x + \frac{x^2}{4}\right]_a^1\\ &= 2\pi\lim_{a\to 0^+}\left[\frac{1}{4} + \frac{a^2}{2}\ln a - \frac{a^2}{4}\right]\\ &= 2\pi\left(\frac{1}{4}\right) = \frac{\pi}{2}\end{aligned}$$

or

$$\begin{aligned}\pi\int_0^\infty x^2 dy &= \pi\lim_{b\to\infty}\int_0^b e^{-2y}\,dy\\ &= \frac{-\pi}{2}\lim_{b\to\infty} e^{-2y}\Big|_0^b\\ &= \frac{-\pi}{2}\lim_{b\to\infty}(e^{-2b} - e^0) = \frac{\pi}{2}\end{aligned}$$

The volume is finite.

1985 BC6

Let f be a function that is defined and twice differentiable for all real numbers x and that has the following properties.

(i) $f(0) = 2$
(ii) $f'(x) > 0$ for all x
(iii) The graph of f is concave up for all $x > 0$ and concave down for all $x < 0$

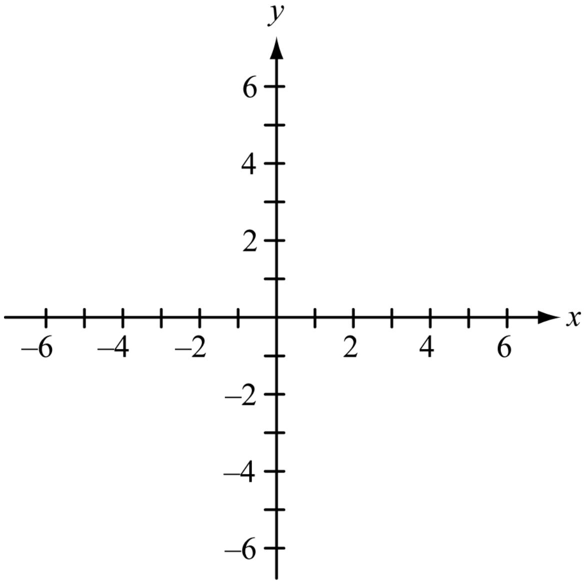
Let g be the function defined by $g(x) = f(x^2)$.

(a) Find $g(0)$.

(b) Find the x-coordinates of all minimum points of g. Justify your answer.

(c) Where is the graph of g concave up? Justify your answer.

(d) Using the information found in parts (a), (b), and (c), sketch a possible graph of g on the axes provided below.



1985 BC6
Solution

(a) $g(0) = f(0) = 2$

(b) $g'(x) = 2xf'(x^2)$

$g'(x) = 0 \Rightarrow x = 0$ or $f'(x^2) = 0$. By (ii), $f'(x^2) > 0$ for all x. Therefore $x = 0$ is the only critical point.

$g'(x) < 0$ for $x < 0$ and $g'(x) > 0$ for $x > 0$ since $f'(x^2) > 0$ for all x. Therefore g is decreasing for $x < 0$ and increasing for $x > 0$. Hence g is a minimum at $x = 0$.

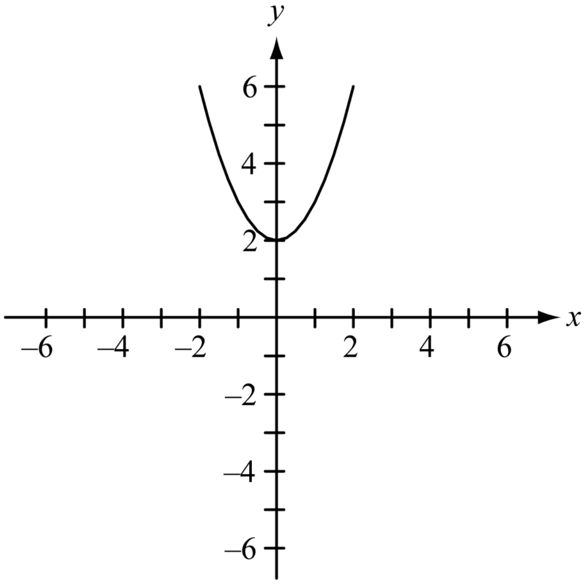
or

Using the second derivative test, $x = 0$ gives a minimum because $g''(0) > 0$ (see part (c) below).

(c) $g''(x) = 2f'(x^2) + 4x^2 f''(x^2)$

By part (ii), $f'(x^2) > 0$ for all x, and by part (iii), $x^2 f''(x^2) \geq 0$ for all x. Therefore $g''(x) > 0$ for all x and hence the graph of g is concave up for all x.

(d) This is one possibility



1986 AB1

Let f be the function defined by $f(x) = 7 - 15x + 9x^2 - x^3$ for all real numbers x.

(a) Find the zeros of f.

(b) Write an equation of the line tangent to the graph of f at $x = 2$.

(c) Find the x-coordinates of all points of inflection of f. Justify your answer.

1986 AB1
Solution

(a) $f(x) = 7 - 15x + 9x^2 - x^3 = -(x-1)^2(x-7)$

The zeros are at $x = 1$ and $x = 7$.

(b) $f'(x) = -15 + 18x - 3x^2$

$f'(2) = -15 + 36 - 12 = 9$

$f(2) = 7 - 30 + 36 - 8 = 5$

The tangent line is $y - 5 = 9(x-2)$ or $y = 9x - 13$.

(c) $f''(x) = 18 - 6x$

$18 - 6x = 0,\ x = 3$

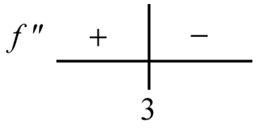
There is a point of inflection at $x = 3$ because

f concave up on $(-\infty, 3)$ and concave down on $(3, \infty)$

or

f'' changes sign from positive to negative at $x = 3$

or



1986 AB2

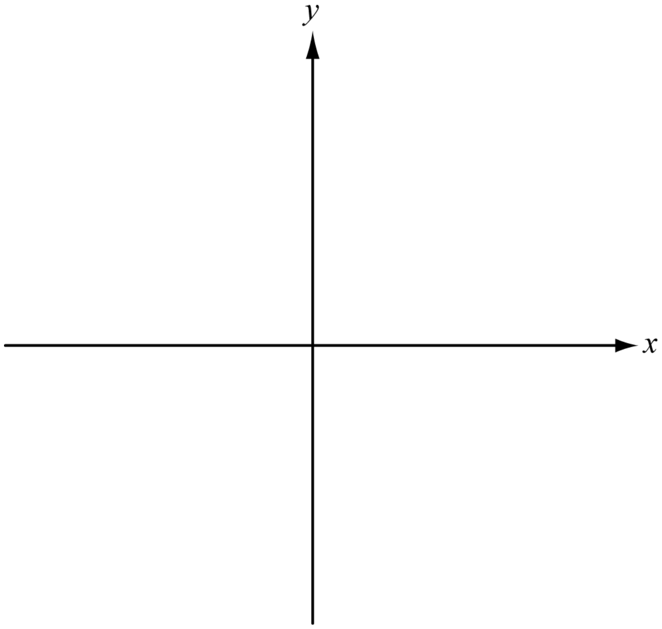
Let f be the function given by $f(x) = \dfrac{9x^2 - 36}{x^2 - 9}$.

(a) Describe the symmetry of the graph of f.

(b) Write an equation for each vertical and each horizontal asymptote of f.

(c) Find the intervals on which f is increasing.

(d) Using the results found in parts (a), (b), and (c), sketch the graph of f on the axes provided below.



1986 AB2
Solution

(a) $f(x) = f(-x)$ indicates symmetry around the y-axis.

(b) Asymptotes:
Vertical: $x = 3,\ x = -3$
Horizontal: $y = 9$

(c) Increasing f:

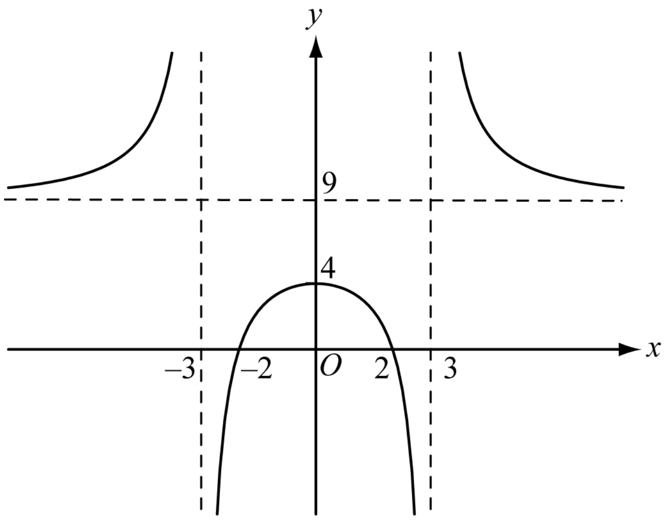
$$f'(x) = \frac{(x^2-9)(18x)-(9x^2-36)(2x)}{(x^2-9)^2} = \frac{-90x}{(x^2-9)^2}$$

Then $f'(x) > 0$ when $-90x > 0$, i.e. when $x < 0$

f'	+	+	−	−
f	inc	inc	dec	dec
	−3	0	3	

The graph of f is increasing on the intervals $(-\infty, -3)$ and $(-3, 0]$.

(d)



1986 AB3/BC1

A particle moves along the x-axis so that at any time $t \geq 1$, its acceleration is given by $a(t) = \frac{1}{t}$. At time $t = 1$, the velocity of the particle is $v(1) = -2$ and its position is $x(1) = 4$.

(a) Find the velocity $v(t)$ for $t \geq 1$.

(b) Find the position $x(t)$ for $t \geq 1$.

(c) What is the position of the particle when it is farthest to the left?

1986 AB3/BC1
Solution

(a) $a(t) = \frac{1}{t},\ t \geq 1$

$v(t) = \int a(t)dt = \ln(t) + C$

$-2 = 0 + C$

$v(t) = \ln(t) - 2$

(b) $x(t) = \int v(t)dt = \int \left(\ln(t) - 2\right) dt = t\ln(t) - t - 2t + C$

$4 = -3 + C$

$7 = C$

$x(t) = t\ln(t) - 3t + 7$

(c) $v(t) = 0 \Rightarrow \ln(t) - 2 = 0$

$t = e^2$

Since $v(1) = -2$, the particle starts out moving to the left. The particle is farthest to the left when $t = e^2$. The position is $x(e^2) = 7 - e^2$.

1986 AB4

Let f be the function defined as follows:

$$f(x) = \begin{cases} |x-1|+2, & \text{for } x<1 \\ ax^2+bx, & \text{for } x \ge 1, \text{ where } a \text{ and } b \text{ are constants.} \end{cases}$$

(a) If $a = 2$ and $b = 3$, is f continuous for all x? Justify your answer.

(b) Describe all values of a and b for which f is a continuous function.

(c) For what values of a and b is f both continuous and differentiable?

1986 AB4
Solution

(a) No, f is not continuous for all x since it is not continuous at $x = 1$.

$\lim_{x\to 1^-} f(x) = 2,\ f(1) = 5$

$\therefore \lim_{x\to 1^-} f(x) \neq f(1)$

or

$\lim_{x\to 1^-} f(x) = 2,\ \lim_{x\to 1^+} f(x) = 5$

$\therefore \lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$

(b) $f(1) = a + b$

or

$\lim_{x\to 1^+} f(x) = a + b$

The function f is continuous when $a + b = 2$.

(c) $f'(x) = \begin{cases} -1 & \text{if } x < 1 \\ 2ax + b & \text{if } x > 1 \end{cases}$

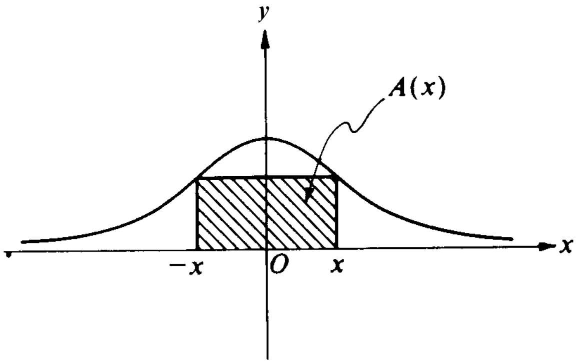
To be continuous and differentiable at $x = 1$, must have

$a + b = 2$

$2a + b = -1$

Therefore $a = -3$ and $b = 5$.

1986 AB5/BC 2



Let $A(x)$ be the area of the rectangle inscribed under the curve $y = e^{-2x^2}$ with vertices at $(-x, 0)$ and $(x, 0)$, $x \geq 0$, as shown in the figure above.

(a) Find $A(1)$.

(b) What is the greatest value of $A(x)$? Justify your answer.

(c) What is the average value of $A(x)$ on the interval $0 \leq x \leq 2$?

1986 AB5/BC2
Solution

(a) $A(1) = 2e^{-2} = \dfrac{2}{e^2}$

(b) $A(x) = 2xe^{-2x^2}$ for $x \ge 0$

$A'(x) = 2e^{-2x^2} + 2xe^{-2x^2}(-4x) = 2e^{-2x^2}(1 - 4x^2) = 0$

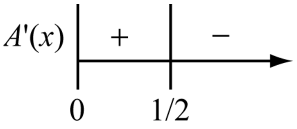
$x = \dfrac{1}{2}$

$A\left(\dfrac{1}{2}\right) = 2 \cdot \dfrac{1}{2} e^{-2 \cdot \frac{1}{4}} = e^{-1/2}$

(i) The sign of $A'(x)$ is determined by the sign of $(1 - 4x^2)$.

$A'(x) > 0$ for $0 < x < \dfrac{1}{2}$ and $A'(x) < 0$ for $\dfrac{1}{2} < x$.

or



Therefore the greatest value of $A(x)$ is $e^{-1/2}$.

or

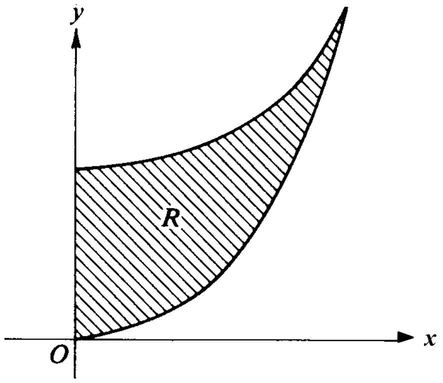
(ii) $A''(x) = -8xe^{-2x^2}\left(1 - 4x^2\right) - 16xe^{-2x^2}$. Therefore $A''\left(\dfrac{1}{2}\right) < 0$ and so $x = \dfrac{1}{2}$ gives a relative maximum. Since there is only one critical value, $x = \dfrac{1}{2}$ also gives the absolute maximum. Therefore the greatest value of $A(x)$ is $e^{-1/2}$.

(c) Average value $= \dfrac{1}{2-0}\displaystyle\int_0^2 A(x)\,dx = \dfrac{1}{2}\int_0^2 2xe^{-2x^2}\,dx$

$$= \int_0^2 xe^{-2x^2}\,dx = -\frac{1}{4}\int_0^{-8} e^u\,du \qquad u = -2x^2;\ du = -4x\,dx$$

$$= -\frac{1}{4}e^u\Big|_0^{-8} = \frac{1}{4}(1 - e^{-8})$$

1986 AB6/BC3



The shaded region R shown in the figure above is enclosed by the graphs of $y = \tan^2 x$, $y = \frac{1}{2}\sec^2 x$, and the y-axis.

(a) Find the area of region R.

(b) Set up, but <u>do not integrate</u>, an integral expression in terms of a single variable for the volume of the solid formed by revolving region R about the x-axis.

1986 AB6/BC3
Solution

(a) $\frac{1}{2}\sec^2 x = \tan^2 x \Rightarrow x = \frac{\pi}{4}$

$$\begin{aligned}\text{Area } &= \int_0^{\pi/4} \left(\frac{1}{2}\sec^2 x - \tan^2 x\right) dx \\ &= \int_0^{\pi/4} \left(1 - \frac{1}{2}\sec^2 x\right) dx \\ &= x - \frac{1}{2}\tan x \Big|_0^{\pi/4} \\ &= \frac{\pi}{4} - \frac{1}{2}\end{aligned}$$

(b) <u>Disks</u>:

$$\text{Volume } = \pi \int_0^{\pi/4} \left(\frac{\sec^4 x}{4} - \tan^4 x\right) dx$$

or

<u>Shells</u>:

$$\text{Volume } = 2\pi \int_0^{1/2} y \arctan\left(\sqrt{y}\right) dy + 2\pi \int_{1/2}^{1} y \left(\arctan \sqrt{y} - \operatorname{arcsec}\sqrt{2y}\right) dy$$

1986 BC4

Given the differential equation $\frac{dy}{dx} = 2y - 5\sin x$:

(a) Find the general solution.

(b) Find the particular solution whose tangent line at $x = 0$ has slope 7.

1986 BC4
Solution

(a) Method 1: Undetermined Coefficient

$y_h = Ce^{2x}$

$y_p = A\sin x + B\cos x$

$\dfrac{dy_p}{dx} = A\cos x - B\sin x$

$A\cos x - B\sin x = 2A\sin x + 2B\cos x - 5\sin x$

$\begin{cases} 2A + B = 5 \\ A - 2B = 0 \end{cases}$

$A = 2,\ B = 1$

$y = Ce^{2x} + 2\sin x + \cos x$

Method 2: Integrating Factor

$\dfrac{dy}{dx} - 2y = -5\sin x$

Integrating factor is $e^{\int -2\,dx} = e^{-2x}$. Multiplying both sides by the integrating factor and antidifferentiating gives

$$ye^{-2x} = -5\int e^{-2x}\sin x\,dx$$

$$\int e^{-2x}\sin x\,dx = -\frac{1}{2}e^{-2x}\sin x + \frac{1}{2}\int e^{-2x}\cos x\,dx$$

$$= -\frac{1}{2}e^{-2x}\sin x - \frac{1}{4}e^{-2x}\cos x - \frac{1}{4}\int e^{-2x}\sin x\,dx$$

or

$$\int e^{-2x}\sin x\,dx = -e^{-2x}\cos x - 2\int e^{-2x}\cos x\,dx$$

$$= -e^{-2x}\cos x - 2e^{-2x}\sin x - 4\int e^{-2x}\cos x\,dx$$

Thus $\int e^{-2x}\sin x\,dx = -\frac{2}{5}e^{-2x}\sin x - \frac{1}{5}e^{-2x}\cos x$, and therefore

$ye^{-2x} = e^{-2x}\cos x + 2e^{-2x}\sin x + C$, or $y = \cos x + 2\sin x + Ce^{2x}$

(b) Using either $7 = 2(Ce^0 + 2\sin 0 + \cos 0) - 5\sin 0$ or $7 = C2e^0 + 2\cos 0 - \sin 0$, we get that $7 = 2C + 2$. Hence $C = \frac{5}{2}$ and so $y = \frac{5}{2}e^{2x} + 2\sin x + \cos x$.

1986 BC5

(a) Find the first four nonzero terms in the Taylor series expansion about $x=0$ for $f(x)=\sqrt{1+x}$.

(b) Use the results found in part (a) to find the first four nonzero terms in the Taylor series expansion about $x=0$ for $g(x)=\sqrt{1+x^3}$.

(c) Find the first four nonzero terms in the Taylor series expansion about $x=0$ for the function h such that $h'(x)=\sqrt{1+x^3}$ and $h(0)=4$.

1986 BC5
Solution

(a) $f(x) = \sqrt{1+x}$ $\qquad f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-1/2}$ $\qquad f'(0) = \frac{1}{2}$

$f''(x) = -\frac{1}{4}(1+x)^{-3/2}$ $\qquad f''(0) = -\frac{1}{4}$

$f'''(x) = \frac{3}{8}(1+x)^{-5/2}$ $\qquad f'''(0) = \frac{3}{8}$

$$T_f(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots.$$

(b) $T_g(x) = 1 + \frac{1}{2}x^3 - \frac{1}{8}x^6 + \frac{1}{16}x^9 + \dots$

(c) Integrating the Taylor series in (b) gives

$T_h(x) = C + x + \frac{1}{8}x^4 - \frac{1}{56}x^7 + \dots$

$h(0) = 4 \Rightarrow C = 4$

$T_h(x) = 4 + x + \frac{1}{8}x^4 - \frac{1}{56}x^7 + \dots$

1986 BC6

For all real numbers x and y, let f be a function such that $f(x+y)=f(x)+f(y)+2xy$ and such that $\lim\limits_{h\to 0}\dfrac{f(h)}{h}=7$.

(a) Find $f(0)$. Justify your answer.

(b) Use the definition of the derivative to find $f'(x)$.

(c) Find $f(x)$.

1986 BC6
Solution

(a) Let $x = y = 0$

Then $f(0+0) = f(0) + f(0) + 2 \cdot 0 \cdot 0$

$$f(0) = 2f(0)$$

$$f(0) = 0$$

(b)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x) + f(h) + 2xh - f(x)}{h} \\ &= \lim_{h\to 0} \left(\frac{f(h)}{h} + 2x \right) \\ &= 7 + 2x \end{aligned}$$

(c)

$$f'(x) = 7 + 2x$$

$$f(x) = 7x + x^2 + C$$

Use $x = 0$:

$0 = f(0) = 0 + C = C$

Therefore $f(x) = 7x + x^2$.

1987 AB1

A particle moves along the x-axis so that its acceleration at any time t is given by $a(t) = 6t - 18$. At time $t = 0$ the velocity of the particle is $v(0) = 24$, and at time $t = 1$, its position is $x(1) = 20$.

(a) Write an expression for the velocity $v(t)$ of the particle at any time t.

(b) For what values of t is the particle at rest?

(c) Write an expression for the position $x(t)$ of the particle at any time t.

(d) Find the total distance traveled by the particle from $t = 1$ to $t = 3$.

1987 AB1
Solution

(a) $a(t) = 6t - 18$

$v(t) = 3t^2 - 18t + C$

$24 = v(0) = C$

$v(t) = 3t^2 - 18t + 24$

(b) $v(t) = 0$ when $3(t^2 - 6t + 8) = 0$

$3(t-4)(t-2) = 0$

The particle is at rest when $t = 2$ and $t = 4$.

(c) $x(t) = t^3 - 9t^2 + 24t + C$

$20 = x(1) = 1 - 9 + 24 + C$

$C = 4$

$x(t) = t^3 - 9t^2 + 24t + 4$

(d) The particle changes direction at $t = 2$.

$x(1) = 20$

$x(2) = 8 - 36 + 48 + 4 = 24$

$x(3) = 27 - 81 + 72 + 4 = 22$

Distance $= x(2) - x(1) + x(2) - x(3) = 4 + 2 = 6$

1987 AB2

Let $f(x)=\sqrt{1-\sin x}$.

(a) What is the domain of f?

(b) Find $f'(x)$.

(c) What is the domain of f'?

(d) Write an equation for the line tangent to the graph of f at $x=0$.

1987 AB2
Solution

(a) $f(x)=\sqrt{1-\sin(x)}$

The domain of f is all real numbers.

(b) $f(x)=(1-\sin(x))^{1/2}$

$$f'(x)=\frac{1}{2}(1-\sin(x))^{-1/2}(-\cos(x))$$

(c) $f'(x)=\dfrac{-\cos(x)}{2\sqrt{1-\sin(x)}}$

We must have $1-\sin(x)>0$ and therefore the domain of f' is all

$$x\neq\frac{\pi}{2}+2k\pi$$

or

$$\mathbb{R}\cap\left\{x\mid x\neq\frac{\pi}{2}+2k\pi,\ k\in I\right\}$$

or

$$x\neq\cdots\frac{-7\pi}{2},\frac{-3\pi}{2},\frac{\pi}{2},\frac{5\pi}{2},\cdots$$

(d) $f'(0)=-\dfrac{1}{2},\ f(0)=1$

The tangent line is

$$y-1=-\frac{1}{2}(x-0)$$

or

$$y=-\frac{1}{2}x+1$$

or

$$x+2y=2$$

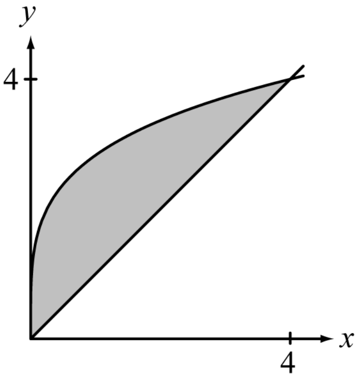
1987 AB3

Let R be the region enclosed by the graphs of $y = (64x)^{\frac{1}{4}}$ and $y = x$.

(a) Find the volume of the solid generated when region R is revolved about the <u>x-axis</u>.

(b) Set up, but <u>do not integrate</u>, an integral expression in terms of a single variable for the volume of the solid generated when region R is revolved about the <u>y-axis</u>.

1987 AB3
Solution



(a) $(64x)^{1/4} = x$

$64x = x^4$

$x = 0$ and $x = 4$

Disks:

$$\text{Volume } = \pi\int_0^4\left((64x)^{1/2} - x^2\right)dx$$

$$= \pi\int_0^4\left(8x^{1/2} - x^2\right)dx$$

$$= \pi\left(8\cdot\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right)\Bigg|_0^4$$

$$= \frac{128\pi}{3} - \frac{64\pi}{3} = \frac{64\pi}{3}$$

or

Shells:

$$\text{Volume } = 2\pi\int_0^4 y\left(y - \frac{y^4}{64}\right)dy$$

$$= 2\pi\left(\frac{y^3}{3} - \frac{y^6}{6\cdot 64}\right)\Bigg|_0^4 = \frac{64\pi}{3}$$

(b) Shells:

$$\text{Volume } = 2\pi\int_0^4 x\left((64x)^{1/4} - x\right)dx$$

or

Disks:

$$\text{Volume } = \pi\int_0^4\left(y^2 - \left(\frac{y^4}{64}\right)^2\right)dy$$

1987 AB4

Let f be the function given by $f(x) = 2\ln(x^2 + 3) - x$ with domain $-3 \le x \le 5$.

(a) Find the x-coordinate of each relative maximum point and each relative minimum point of f. Justify your answer.

(b) Find the x-coordinate of each inflection point of f.

(c) Find the absolute maximum value of $f(x)$.

1987 AB4
Solution

(a) $f(x) = 2\ln(x^2+3) - x$

$$f'(x) = 2\cdot\frac{2x}{x^2+3} - 1 = \frac{-(x-3)(x-1)}{x^2+3}$$

f'	$-$	$+$	$-$
f	dec	inc	dec

-3 $\quad$ 1 $\quad$ 3 $\quad$ 5

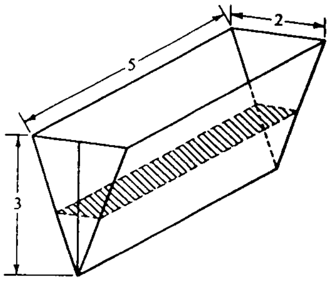
There is a relative minimum at $x = 1$ because f' changes from negative to positive.
There is a relative maximum at $x = 3$ because f' changes from positive to negative.

(b) $$f''(x) = \frac{4(x^2+3) - 4x\cdot 2x}{(x^2+3)^2} = \frac{12-4x^2}{(x^2+3)^2} = \frac{4(3-x^2)}{(x^2+3)^2}$$

The inflection points are at $x = \sqrt{3}$ and $x = -\sqrt{3}$.

(c) $f(-3) = 2\ln 12 + 3$
$f(3) = 2\ln 12 - 3$
The absolute maximum value is $2\ln 12 + 3$.

1987 AB5



The trough shown in the figure above is 5 feet long, and its vertical cross sections are inverted isosceles triangles with base 2 feet and height 3 feet. Water is being siphoned out of the trough at the rate of 2 cubic feet per minute. At any time t, let h be the depth and V be the volume of water in the trough.

(a) Find the volume of water in the trough when it is full.

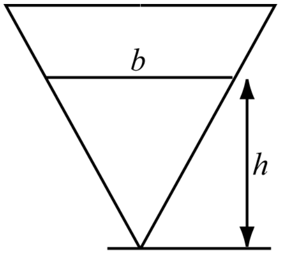
(b) What is the rate of change in h at the instant when the trough is $\frac{1}{4}$ full by volume?

(c) What is the rate of change in the area of the surface of the water (shaded in the figure) at the instant when the trough is $\frac{1}{4}$ full by volume?

1987 AB5
Solution

(a) Volume = $\frac{1}{2}\cdot 2\cdot 3\cdot 5 = 15$

(b) $V = 5\cdot\frac{1}{2}bh$



By similar triangles, $\frac{b}{h} = \frac{2}{3}$. Therefore $V = \frac{5}{3}h^2$ and

$\frac{dV}{dt} = \frac{10}{3}h\frac{dh}{dt}$.

When the trough is $\frac{1}{4}$ full by volume, $\frac{15}{4} = \frac{5}{3}h^2$ and

therefore $h = \frac{3}{2}$.

At this instant $-2 = \frac{10}{3}\cdot\frac{3}{2}\cdot\frac{dh}{dt}$ and thus $\frac{dh}{dt} = -\frac{2}{5}$.

(c) $A = 5b = \frac{10}{3}h$

$\frac{dA}{dt} = \frac{10}{3}\frac{dh}{dt}$

$\frac{dA}{dt} = \frac{10}{3}\cdot\frac{-2}{5} = \frac{-4}{3}$

1987 AB6

Let f be a function such that $f(x) < 1$ and $f'(x) < 0$ for all x.

(a) Suppose that $f(b) = 0$ and $a < b < c$. Write an expression involving integrals for the area of the region enclosed by the graph of f, the lines $x = a$ and $x = c$, and the x-axis.

(b) Determine whether $g(x) = \dfrac{1}{f(x) - 1}$ is increasing or decreasing. Justify your answer.

(c) Let h be a differentiable function such that $h'(x) < 0$ for all x. Determine whether $F(x) = h(f(x))$ is increasing or decreasing. Justify your answer.

1987 AB6
Solution

(a) Area = $\int_a^b f(x)dx - \int_b^c f(x)dx$ or Area = $\int_a^c |f(x)|dx$

(b) $g'(x) = \dfrac{-f'(x)}{(f(x)-1)^2}$

$f'(x) < 0$ and $(f(x)-1)^2 > 0 \Rightarrow g'(x) > 0$ for all x. Therefore g is increasing.

It is possible to give a non-calculus argument. Since $f'(x) < 0$ for all x, the function f is decreasing for all x. Therefore the function $f(x)-1$ is decreasing for all x. Since $f(x)-1 < 0$ for all x, it follows that $\dfrac{1}{f(x)-1} = g(x)$ is increasing for all x.

(c) $F'(x) = h'(f(x)) \cdot f'(x)$
$h' < 0$ and $f' < 0 \Rightarrow F' > 0$ for all x.
Therefore F is increasing.

Non-calculus argument:
$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ since f is decreasing.
Therefore $h\big(f(x_1)\big) < h\big(f(x_2)\big)$ since h is decreasing.
So $x_1 < x_2 \Rightarrow F(x_1) < F(x_2)$. Hence F is increasing.

1987 BC1

At any time $t \geq 0$, in days, the rate of growth of a bacteria population is given by $y' = ky$, where k is a constant and y is the number of bacteria present. The initial population is 1,000 and the population triples during the first 5 days.

(a) Write an expression for y at any time $t \geq 0$.

(b) By what factor will the population have increased in the first 10 days?

(c) At what time t, in days, will the population have increased by a factor of 6?

1987 BC1
Solution

(a) $y = Ae^{kt}$

$y(0) = 1000 = A$

$3 = e^{k \cdot 5}$

$k = \dfrac{\ln 3}{5}$

Therefore $y = 1000e^{\frac{t \ln 3}{5}}$ or $y = 1000 \cdot 3^{t/5}$

(b) $y(10) = 1000e^{\frac{10 \ln 3}{5}} = 1000 \cdot 3^2$

Therefore the population will have increased by a factor of 9.

(c) $6000 = 1000e^{\frac{t \ln 3}{5}}$

$6 = e^{\frac{t \ln 3}{5}}$

$t = \dfrac{5 \ln 6}{\ln 3}$

1987 BC2

Consider the curve given by the equation $y^3 + 3x^2 y + 13 = 0$.

(a) Find $\frac{dy}{dx}$.

(b) Write an equation for the line tangent to the curve at the point $(2, -1)$.

(c) Find the minimum y-coordinate of any point on the curve. Justify your answer.

1987 BC2
Solution

(a) $3y^2y' + 3x^2y' + 6xy = 0$

$$y' = -\frac{6xy}{3x^2 + 3y^2} = -\frac{2xy}{x^2 + y^2}$$

(b) At the point $(2,-1)$, $y' = \frac{-(2)(2)(-1)}{4+1} = \frac{4}{5}$

The equation of the tangent line is $y + 1 = \frac{4}{5}(x-2)$ or $y = \frac{4}{5}x - \frac{13}{5}$.

(c) $y' = \frac{-2xy}{x^2 + y^2} = 0 \Rightarrow x = 0$ or $y = 0$

Since y cannot be 0 for any point on the curve, we must have $x = 0$. We claim that this gives the minimum y-value on the curve. At $x = 0$, $y = -\sqrt[3]{13}$.

$y(y^2 + 3x^2) = -13 \Rightarrow y < 0$. Therefore $y' < 0$ for $x < 0$ and $y' > 0$ for $x > 0$. Thus $x = 0$ does give the minimum value of $y = -\sqrt[3]{13}$.

Non-calculus argument: $y(y^2 + 3x^2) = -13 \Rightarrow y < 0$. Therefore $y^3 + 13 = -3x^2y \geq 0$ for all points on the curve. Thus $y \geq -\sqrt[3]{13}$ for all points on the curve. But $y = -\sqrt[3]{13}$ when $x = 0$, thus $y = -\sqrt[3]{13}$ is the minimum.

1987 BC3

Let R be the region enclosed by the graph of $y = \ln x$, the line $x = 3$, and the x-axis.

(a) Find the area of region R.

(b) Find the volume of the solid generated by revolving region R about the x-axis.

(c) Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid generated by revolving region R about the line $x = 3$.

1987 BC3
Solution

(a) Area $= \int_1^3 \ln x\, dx$

$= (x\ln x - x)\Big|_1^3$

$= 3\ln 3 - 2$

(b) <u>Disks</u>:

Volume $= \pi\int_1^3 (\ln x)^2\, dx$

$= \pi\left(x(\ln x)^2\Big|_1^3 - \int_1^3 2\ln x\, dx \right)$

$= \pi\left(x(\ln x)^2 - 2x\ln x + 2x \right)\Big|_1^3$

$= \pi\left(3(\ln 3)^2 - 6\ln 3 + 4 \right)$

or

<u>Shells</u>:

Volume $= 2\pi\int_0^{\ln 3} y(3 - e^y)\, dy$

$= 2\pi\left(\frac{3}{2}y^2 - ye^y + e^y \right)\Bigg|_0^{\ln 3}$

$= 2\pi\left(\left(\frac{3}{2}(\ln 3)^2 - \ln 3 \cdot e^{\ln 3} + e^{\ln 3} \right) - 1 \right)$

$= 2\pi\left(\frac{3}{2}(\ln y)^2 - 3\ln 3 + 2 \right)$

(c) <u>Shells</u>:

Volume $= 2\pi\int_1^3 (3 - x)\ln x\, dx$

or

<u>Disks:</u>

Volume $= \pi\int_0^{\ln 3} (3 - e^y)^2\, dy$

1987 BC4

(a) Find the first five terms in the Taylor series about $x = 0$ for $f(x) = \frac{1}{1-2x}$.

(b) Find the interval of convergence for the series in part (a).

(c) Use partial fractions and the result from part (a) to find the first five terms in the Taylor series about $x = 0$ for $g(x) = \frac{1}{(1-2x)(1-x)}$.

1987 BC4
Solution

(a) Using geometric series, $\frac{1}{1-2x} \approx 1+2x+(2x)^2+(2x)^3+(2x)^4$

or

$f^{(n)}(x) = 2^n \cdot n! \cdot (1-2x)^{-(n+1)}$
$f^{(n)}(0) = 2^n \cdot n!$
Therefore $\frac{1}{1-2x} \approx 1+2x+4x^2+8x^3+16x^4$

(b) The Taylor series for $\frac{1}{1-2x}$ is a geometric series and thus converges for $|2x|<1$ or $|x|<\frac{1}{2}$.

Alternatively, can use the ratio test.

$$\left|\frac{2^{n+1}x^{n+1}}{2^n x^n}\right| = |2x| < 1 \Rightarrow |x| < \frac{1}{2}$$

Checking the endpoints,
At $x=-\frac{1}{2}$, the series is $1-1+1-1+\ldots$ which diverges.
At $x=\frac{1}{2}$, the series is $1+1+1+1+\ldots$ which diverges.
Therefore the interval of convergence is $|x|<\frac{1}{2}$.

(c) $$\frac{1}{(1-2x)(1-x)} = \frac{2}{1-2x} - \frac{1}{1-x}$$

$$\frac{1}{(1-2x)(1-x)} \approx (2+4x+8x^2+16x^3+32x^4)-(1+x+x^2+x^3+x^4)$$

$$= 1+3x+7x^2+15x^3+31x^4$$

1987 BC5

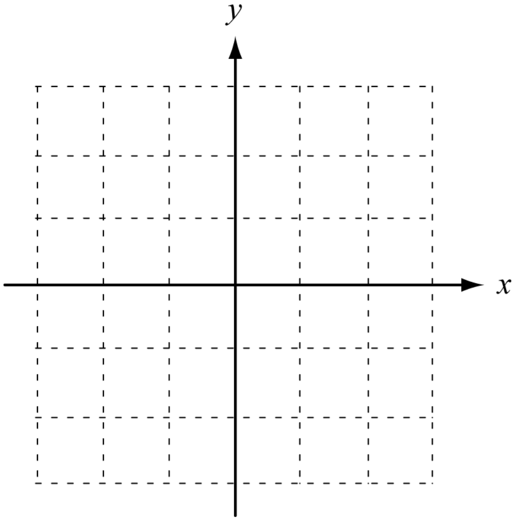
The position of a particle moving in the xy-plane at any time t, $0 \le t \le 2\pi$, is given by the parametric equations $x = \sin t$ and $y = \cos(2t)$.

(a) Find the velocity vector for the particle at any time t, $0 \le t \le 2\pi$.

(b) For what values of t is the particle at rest?

(c) Write an equation for the path of the particle in terms of x and y that does not involve trigonometric functions.

(d) Sketch the path of the particle in the xy-plane below.



1987 BC5
Solution

(a) $x = \sin(t), y = \cos(2t)$

$\frac{dx}{dt} = \cos(t), \frac{dy}{dt} = -2\sin(2t)$

$\vec{v} = (\cos(t), -2\sin(2t))$

or

$\vec{\mathrm{v}} = \cos(t)\vec{i} + (-2\sin(2t))\vec{j}$

(b) $\vec{v} = 0 \Rightarrow \cos(t) = 0$ and $-2\sin(2t) = 0$

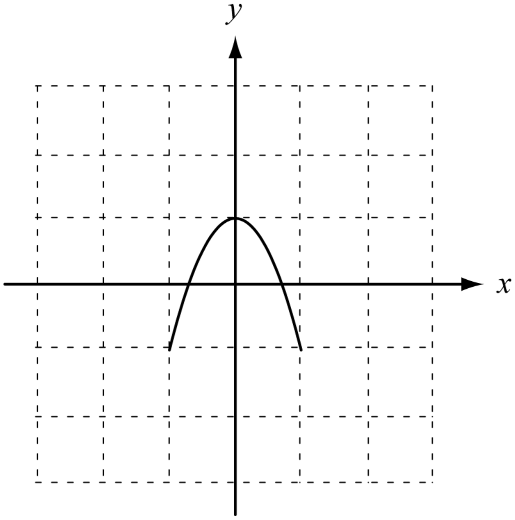
Therefore $\cos(t) = 0$ and $-4\sin(t)\cos(t) = 0$. The only choice is $\cos(t) = 0$.

Therefore the particle is at rest when $t = \frac{\pi}{2}, \frac{3\pi}{2}$.

(c) $x^2 = \sin^2(t),\ y = \cos^2(t) = 1 - 2\sin^2(t)$

$y = 1 - 2x^2$

(d)



1987 BC6

Let f be a continuous function with domain $x > 0$ and let F be the function given by $F(x) = \int_1^x f(t)\,dt$ for $x > 0$. Suppose that $F(ab) = F(a) + f(b)$ for all $a > 0$ and $b > 0$ and that $F'(1) = 3$.

(a) Find $f(1)$.

(b) Prove that $aF'(ax) = F'(x)$ for every positive constant a.

(c) Use the results from parts (a) and (b) to find $f(x)$. Justify your answer.

1987 BC6
Solution

(a) $F'(x) = f(x)$

$F'(1) = f(1) = 3$

(b) $F(ax) = F(a) + F(x)$

$$aF'(ax) = \frac{d}{dx}F(ax) = \frac{d}{dx}\big(F(a) + F(x)\big) = F'(x)$$

(c) For all $a > 0$, $aF'(ax) = F'(x)$. Let $x = 1$.

$aF'(a) = F'(1) = 3$

$$F'(a) = \frac{3}{a}$$

Replace a with x:

$$F'(x) = \frac{3}{x}$$

$$f(x) = F'(x) = \frac{3}{x}$$

1988 AB1

Let f be the function given by $f(x) = \sqrt{x^4 - 16x^2}$.

(a) Find the domain of f.

(b) Describe the symmetry, if any, of the graph of f.

(c) Find $f'(x)$.

(d) Find the slope of the line <u>normal</u> to the graph of f at $x = 5$.

1988 AB1
Solution

(a) $x^4 - 16x^2 \geq 0$

$x^2(x^2 - 16) \geq 0$

$x^2 \geq 16$ or $x = 0$

The domain of f is all x satisfying $|x| \geq 4$ or $x = 0$.

(b) The graph of f is symmetric about the y-axis because $f(-x) = f(x)$.

(c)
$$f'(x) = \frac{1}{2}(x^4 - 16x^2)^{-1/2}(4x^3 - 32x)$$
$$= \frac{2x(x^2 - 8)}{|x|\sqrt{x^2 - 16}}$$

(d)
$$f'(5) = \frac{2 \cdot 125 - 16 \cdot 5}{\sqrt{625 - 16 \cdot 25}}$$
$$= \frac{170}{15}$$
$$= \frac{34}{3}$$

Therefore the slope of the normal line is $m = -\dfrac{3}{34}$.

1988 AB2

A particle moves along the x-axis so that its velocity at any time $t \geq 0$ is given by $v(t) = 1 - \sin(2\pi t)$.

(a) Find the acceleration $a(t)$ of the particle at any time t.

(b) Find all values of t, $0 \leq t \leq 2$, for which the particle is at rest.

(c) Find the position $x(t)$ of the particle at any time t if $x(0) = 0$.

1988 AB2
Solution

(a) $a(t) = v'(t)$

$= -2\pi\cos(2\pi t)$

(b) $v(t) = 0$ gives $1 - \sin(2\pi t) = 0$ or $1 = \sin(2\pi t)$. Therefore $2\pi t = \frac{\pi}{2} + 2k\pi$ where $k = 0, \pm 1, \pm 2, \ldots$, and $0 \le t \le 2$. The two solutions are $t = \frac{1}{4}, \frac{5}{4}$.

(c) $x(t) = \int v(t)\,dt = \int \left(1 - \sin(2\pi t)\right) dt = t + \frac{1}{2\pi}\cos(2\pi t) + C$

$x(0) = 0 \Rightarrow 0 = 0 + \frac{\cos(0)}{2\pi} + C$

$C = -\frac{1}{2\pi}$

Therefore $x(t) = t + \frac{1}{2\pi}\cos(2\pi t) - \frac{1}{2\pi}$

1988 AB3

Let R be the region in the first quadrant enclosed by the hyperbola $x^2 - y^2 = 9$, the x-axis, and the line $x = 5$.

(a) Find the volume of the solid generated by revolving R about the $\underline{x\text{-axis}}$.

(b) Set up, <u>but do not integrate</u>, an integral expression in terms of a single variable for the volume of the solid generated when R is revolved about the line $x = -1$.

1988 AB3
Solution

(a) Disks:

$$\text{Volume} = \pi\int_3^5 (x^2 - 9)\,dx$$

$$= \pi\left(\frac{1}{3}x^3 - 9x\right)\Bigg|_3^5$$

$$= \pi\left(\left(\frac{125}{3} - 45\right) - (9 - 27)\right) = \frac{44}{3}\pi$$

or

Shells:

$$\text{Volume} = 2\pi\int_0^4\left(5 - \sqrt{9 + y^2}\right)y\,dy$$

$$= 2\pi\left(\frac{5}{2}y^2 - \frac{1}{3}(9 + y^2)^{3/2}\right)\Bigg|_0^4$$

$$= 2\pi\left(40 - \frac{125}{3} + \frac{27}{3}\right) = \frac{44}{3}\pi$$

(b) Shells:

$$\text{Volume} = 2\pi\int_3^5 (x + 1)y\,dx$$

$$= 2\pi\int_3^5 (x + 1)\sqrt{x^2 - 9}\,dx$$

or

Disks:

$$\text{Volume} = \pi\int_0^4\left(36 - (x + 1)^2\right)dy$$

$$= \pi\int_0^4\left(36 - (\sqrt{9 + y^2} + 1)^2\right)dy$$

1988 AB4

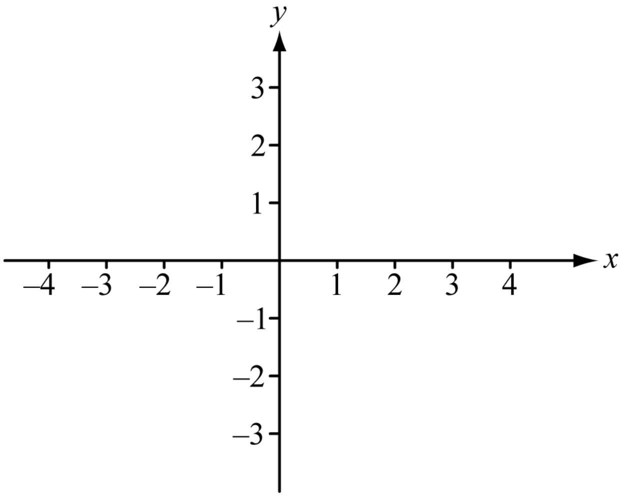
Let f be the function defined by $f(x) = 2xe^{-x}$ for all real numbers x.

(a) Write an equation of the horizontal asymptote for the graph of f.

(b) Find the x-coordinate of each critical point of f. For each such x, determine whether $f(x)$ is a relative maximum, a relative minimum, or neither.

(c) For what values of x is the graph of f concave down?

(d) Using the results found in parts (a), (b), and (c), sketch the graph of $y = f(x)$ on the axes provided below.



1988 AB4
Solution

(a) $y = 0$

(b) $f'(x) = 2\left(-xe^{-x} + e^{-x}\right) = 2e^{-x}(1-x)$

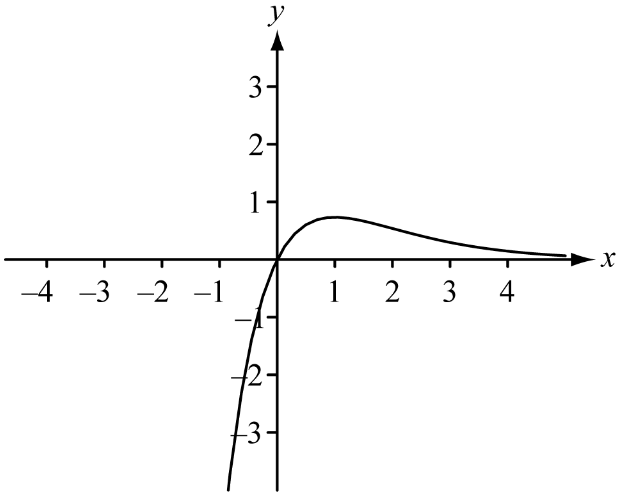
There is a critical point at $x = 1$ where $f(x)$ has a relative maximum since $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$.

(c) $f''(x) = 2e^{-x}(-1) + (-2e^{-x})(1-x) = 2e^{-x}(x-2)$

The graph of f is concave down when:

$$2e^{-x}(x-2) < 0$$
$$x - 2 < 0$$
$$x < 2$$

(d)



1988 AB5

Let R be the region in the first quadrant under the graph of $y = \dfrac{x}{x^2 + 2}$ for $0 \le x \le \sqrt{6}$?

(a) Find the area of R.

(b) If the line $x = k$ divides R into two regions of equal area, what is the value of k?

(c) What is the average value of $y = \dfrac{x}{x^2 + 2}$ on the interval $0 \le x \le \sqrt{6}$?

1988 AB5
Solution

(a) Area $= \int_0^{\sqrt{6}} \frac{x}{x^2+2}\,dx$

$$= \frac{1}{2}\ln(x^2+2)\Big|_0^{\sqrt{6}}$$

$$= \frac{1}{2}\ln 8 - \frac{1}{2}\ln 2 = \ln 2$$

(b) $\frac{1}{2}\ln 2 = \int_0^{k} \frac{x}{x^2+2}\,dx$

$$= \frac{1}{2}\ln(x^2+2)\Big|_0^{k}$$

$$= \frac{1}{2}\ln(k^2+2) - \frac{1}{2}\ln 2$$

$$\frac{1}{2}\ln(k^2+2) = \frac{1}{2}\ln 2 + \frac{1}{2}\ln 2 = \ln 2$$

$$\ln(k^2+2) = \ln 4$$

Therefore $k^2+2=4$ and so $k=\sqrt{2}$.

(c) Average value $= \frac{1}{\sqrt{6}-0}\int_0^{\sqrt{6}} \frac{x}{x^2+2}\,dx = \frac{1}{\sqrt{6}}\ln 2$

1988 AB6

Let f be a differentiable function, defined for all real numbers x, with the following properties.

(i) $f'(x) = ax^2 + bx$

(ii) $f'(1) = 6$ and $f''(1) = 18$

(iii) $\int_1^2 f(x)\,dx = 18$

Find $f(x)$. Show your work.

1988 AB6
Solution

Differentiating the expression in (i) gives $f''(x) = 2ax + b$

Let $x = 1$. Then from (ii),

$$a + b = f'(1) = 6$$
$$2a + b = f''(1) = 18$$

Solving these two equations gives $a = 12$ and $b = -6$. Therefore $f'(x) = 12x^2 - 6x$ and hence

$$f(x) = 4x^3 - 3x^2 + C$$

Using (iii) gives

$$\begin{aligned} 18 &= \int_1^2 (4x^3 - 3x^2 + C)\,dx \\ &= (x^4 - x^3 + Cx)\Big|_1^2 \\ &= (16 - 8 + 2C) - (1 - 1 + C) = 8 + C \end{aligned}$$

Hence $C = 10$ and $f(x) = 4x^3 - 3x^2 + 10$.

1988 BC1

Let f be the function defined by $f(x) = (x^2 - 3)e^x$ for all real numbers x.

(a) For what values of x is f increasing?

(b) Find the x-coordinate of each point of inflection of f.

(c) Find the x- and y-coordinates of the point, if any, where $f(x)$ attains its absolute minimum.

1988 BC1
Solution

(a) $f'(x) = (x^2 - 3)e^x + 2xe^x = e^x(x+3)(x-1)$

$f'(x) = 0$ for $x = -3, 1$.

f'	+	−	+
f	inc	dec	inc

$-3 \qquad 1$

Therefore f is increasing for $x < -3$ and $x > 1$.

(b) $f''(x) = (x^2 - 3 + 2x)e^x + (2x+2)e^x = e^x(x^2 + 4x - 1)$

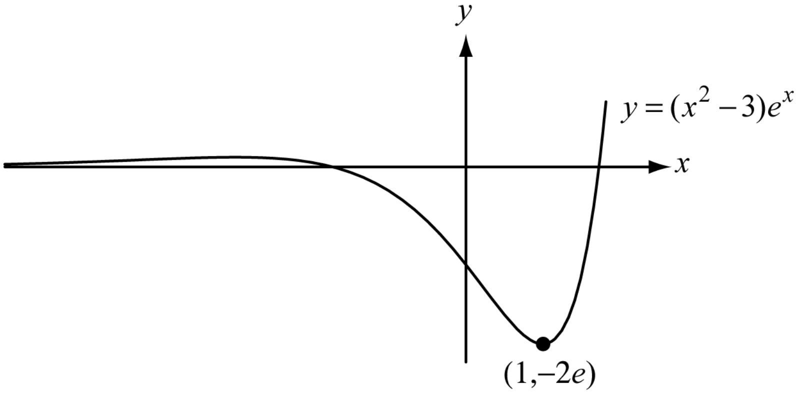
$f''(x) = 0$ for $x = \dfrac{-4 \pm \sqrt{20}}{2}$

The points of inflection occur at $x = -2 \pm \sqrt{5}$.

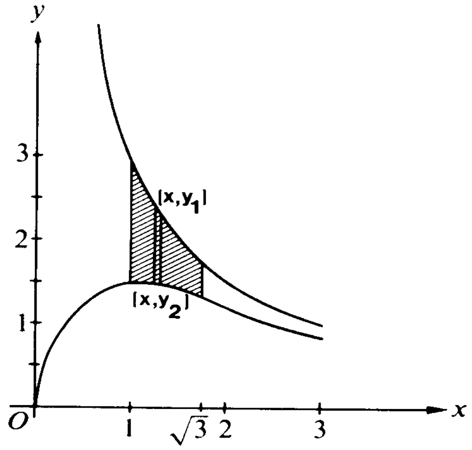
(c) $f(x)$ has a relative minimum at $x = 1$; $f(1) = -2e$.

$f(x)$ has a relative maximum at $x = -3$ and $\lim\limits_{x \to -\infty} f(x) = 0$

So, f has an absolute minimum at $x = 1$, $y = -2e$.



1988 BC2



Let R be the shaded region between the graphs of $y = \frac{3}{x}$ and $y = \frac{3x}{x^2+1}$ from $x = 1$ to $x = \sqrt{3}$, as shown in the figure above.

(a) Find the area of R.

(b) Find the volume of the solid generated by revolving R about the y-axis.

1988 BC2
Solution

(a) Area $= \displaystyle\int_1^{\sqrt{3}} \left(\frac{3}{x} - \frac{3x}{x^2+1}\right) dx$

$$= \left(3\ln x - \frac{3}{2}\ln\left(x^2+1\right)\right)\Bigg|_1^{\sqrt{3}}$$

$$= \left(3\ln\sqrt{3} - \frac{3}{2}\ln 4\right) - \left(0 - \frac{3}{2}\ln 2\right)$$

$$= \frac{3}{2}\ln 3 - \frac{3}{2}\ln 4 + \frac{3}{2}\ln 2$$

$$= \frac{3}{2}\ln\frac{3}{2}$$

(b) Volume $= 2\pi \displaystyle\int_1^{\sqrt{3}} x\left(\frac{3}{x} - \frac{3x}{x^2+1}\right) dx$

$$= 2\pi \int_1^{\sqrt{3}} \frac{3}{x^2+1}\, dx$$

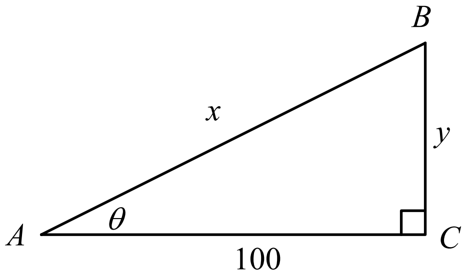
$$= (6\pi \arctan x)\Big|_1^{\sqrt{3}}$$

$$= 6\pi(\arctan\sqrt{3} - \arctan 1)$$

$$= 6\pi\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$= \frac{\pi^2}{2}$$

1988 BC3



The figure above represents an observer at point A watching balloon B as it rises from point C. The balloon is rising at a constant rate of 3 meters per second and the observer is 100 meters from point C.

(a) Find the rate of change in x at the instant when $y = 50$.

(b) Find the rate of change in the area of right triangle BCA at the instant when $y = 50$.

(c) Find the rate of change in θ at the instant when $y = 50$.

1988 BC3
Solution

(a) $x^2 = y^2 + 100^2$

$$2x\frac{dx}{dt} = 2y\frac{dy}{dt}$$

At $y = 50$, $x = 50\sqrt{5}$ and $\frac{dx}{dt} = \frac{3 \cdot 50}{50\sqrt{5}} = \frac{3\sqrt{5}}{5}$ m/s

Explicitly: $x = \sqrt{y^2 + 100^2} \Rightarrow \frac{dx}{dt} = \frac{2y}{2\sqrt{y^2 + 100^2}}\frac{dy}{dt}$

$$= \frac{50}{\sqrt{12500}}(3)$$

$$= \frac{3\sqrt{5}}{5}\text{ m/s}$$

(b) $A = \frac{100y}{2} = 50y$

$$\frac{dA}{dt} = 50\frac{dy}{dt} = 50 \cdot 3 = 150 \text{ m/s}$$

(c) $\tan\theta = \frac{y}{100}$

$$\sec^2\theta\frac{d\theta}{dt} = \frac{1}{100}\frac{dy}{dt} = \frac{3}{100}$$

$$\frac{d\theta}{dt} = \frac{3}{100}\cos^2\theta$$

At $y = 50$, $\cos\theta = \frac{100}{50\sqrt{5}}$ and therefore $\frac{d\theta}{dt} = \frac{3}{100}\left(\frac{2}{\sqrt{5}}\right)^2 = \frac{3}{125}$ radians/sec.

1988 BC4

Determine all values of x for which the series $\sum_{k=0}^{\infty} \frac{2^k x^k}{\ln(k+2)}$ converges. Justify your answer.

1988 BC4
Solution

$$\lim_{k\to\infty}\left|\frac{\dfrac{2^{k+1}x^{k+1}}{\ln(k+3)}}{\dfrac{2^k x^k}{\ln(k+2)}}\right| = \lim_{k\to\infty}|2x|\frac{\ln(k+2)}{\ln(k+3)}$$

By L'Hôpital's Rule,

$$\lim_{k\to\infty}\frac{\ln(k+2)}{\ln(k+3)} = \lim_{k\to\infty}\frac{\dfrac{1}{k+2}}{\dfrac{1}{k+3}} = \lim_{k\to\infty}\frac{k+3}{k+2} = 1 \text{ and so } \lim_{k\to\infty}|2x|\frac{\ln(k+2)}{\ln(k+3)} = |2x|.$$

$|2x| < 1 \Leftrightarrow |x| < \frac{1}{2}$

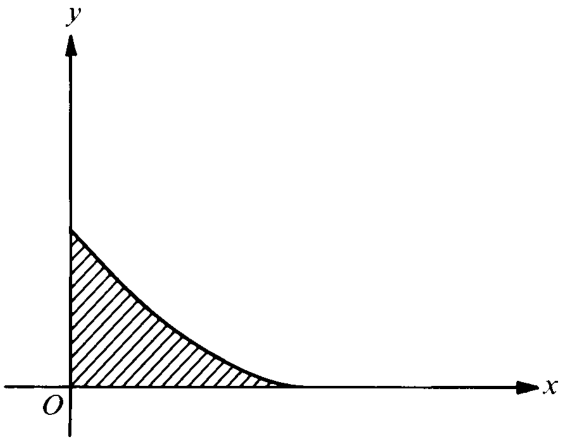
Therefore the series converges at least for $-\frac{1}{2} < x < \frac{1}{2}$. Now check the endpoints.

At $x = \frac{1}{2}$, the series becomes $\sum_{k=0}^{\infty}\frac{1}{\ln(k+2)}$ which diverges by comparison with the harmonic series $\sum_{k=0}^{\infty}\frac{1}{k+2}$.

At $x = -\frac{1}{2}$, the series becomes $\sum_{k=0}^{\infty}\frac{(-1)^k}{\ln(k+2)}$ which converges by the alternating series test.

Therefore the series converges for $-\frac{1}{2} \le x < \frac{1}{2}$

1988 BC5



The base of a solid S is the shaded region in the first quadrant enclosed by the coordinate axes and the graph of $y = 1 - \sin x$, as shown in the figure above. For each x, the cross section of S perpendicular to the x-axis at the point $(x, 0)$ is an isosceles right triangle whose hypotenuse lies in the xy-plane.

(a) Find the area of the triangle as a function of x.

(b) Find the volume of S.

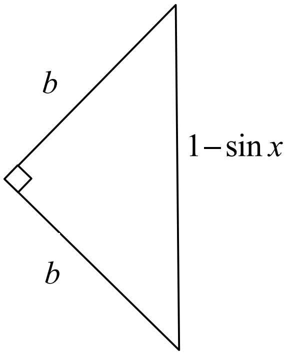
1988 BC5
Solution

(a) $2b^2 = (1-\sin x)^2$

$b^2 = \frac{1}{2}(1-\sin x)^2$

$\text{Area} = \frac{1}{2}b^2$

$A(x) = \frac{1}{4}(1-\sin x)^2$



(b)
$$\begin{aligned}\text{Volume} &= \int_0^{\pi/2} \frac{1}{4}(1-\sin x)^2\,dx \\ &= \frac{1}{4}\int_0^{\pi/2}\left(1-2\sin x+\frac{1}{2}(1-\cos 2x)\right)dx \\ &= \frac{1}{4}\left(x+2\cos x+\frac{1}{2}x-\frac{1}{4}\sin 2x\right)\Bigg|_0^{\pi/2} \\ &= \frac{1}{4}\left(\left(\frac{\pi}{2}+\frac{\pi}{4}\right)-2\right) \\ &= \frac{3\pi}{16}-\frac{1}{2}\end{aligned}$$

1988 BC6

Let f be a differentiable function defined for all $x \geq 0$ such that $f(0) = 5$ and $f(3) = -1$. Suppose that for any number $b > 0$, the average value of $f(x)$ on the interval $0 \leq x \leq b$ is $\dfrac{f(0) + f(b)}{2}$.

(a) Find $\int_0^3 f(x)\,dx$.

(b) Prove that $f'(x) = \dfrac{f(x) - 5}{x}$ for all $x > 0$.

(c) Using part (b), find $f(x)$.

1988 BC6
Solution

(a) $\frac{1}{3}\int_0^3 f(x)\,dx = \frac{f(0)+f(3)}{2} = \frac{5-1}{2} = 2$

Therefore $\int_0^3 f(x)\,dx = 6$

(b) $\frac{1}{x}\int_0^x f(t)\,dt = \frac{f(0)+f(x)}{2}$, for all $x > 0$

$$\int_0^x f(t)\,dt = \frac{5}{2}x + \frac{1}{2}x\,f(x)$$

Differentiating both sides with respect to x gives

$$f(x) = \frac{5}{2} + \frac{1}{2}f(x) + \frac{1}{2}x\,f'(x)$$
$$2f(x) = 5 + f(x) + x\,f'(x)$$
$$f'(x) = \frac{f(x)-5}{x}, \text{ for all } x > 0$$

(c) $\frac{dy}{dx} = \frac{y-5}{x}$. Separating variables gives $\frac{dy}{y-5} = \frac{dx}{x}$.

$$\ln(y-5) = \ln(x) + \ln C$$
$$y - 5 = Cx$$

Since $f(3) = -1$, $C = -2$ and so $f(x) = 5 - 2x$.

or

$$y' - \left(\frac{1}{x}\right)y = -\frac{5}{x}$$

The integrating factor is $e^{\int \frac{-1}{x}dx} = \frac{1}{x}$. Multiplying both sides by this factor and antidifferentiating gives

$$y \cdot \frac{1}{x} = \int -\frac{5}{x^2}\,dx = \frac{5}{x} + C$$

$$y = 5 + Cx$$

Since $f(3) = -1$, $C = -2$ and so $f(x) = 5 - 2x$.