1998 AP Calculus BC: Section I, Part A

55 Minutes—No Calculator

Note: Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

1. What are all values of x for which the function f defined by $f(x) = x^3 + 3x^2 - 9x + 7$ is increasing?

   (A) $-3 < x < 1$
   (B) $-1 < x < 1$
   (C) $x < -3$ or $x > 1$
   (D) $x < -1$ or $x > 3$
   (E) All real numbers

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2. In the xy-plane, the graph of the parametric equations $x = 5t + 2$ and $y = 3t$, for $-3 \le t \le 3$, is a line segment with slope

   (A) $\frac{3}{5}$ (B) $\frac{5}{3}$ (C) 3 (D) 5 (E) 13

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3. The slope of the line tangent to the curve $y^2 + (xy+1)^3 = 0$ at $(2,-1)$ is

   (A) $-\frac{3}{2}$ (B) $-\frac{3}{4}$ (C) 0 (D) $\frac{3}{4}$ (E) $\frac{3}{2}$

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4. $\int \frac{1}{x^2 - 6x + 8}\,dx =$

   (A) $\frac{1}{2}\ln\left|\frac{x-4}{x-2}\right| + C$

   (B) $\frac{1}{2}\ln\left|\frac{x-2}{x-4}\right| + C$

   (C) $\frac{1}{2}\ln\left|(x-2)(x-4)\right| + C$

   (D) $\frac{1}{2}\ln\left|(x-4)(x+2)\right| + C$

   (E) $\ln\left|(x-2)(x-4)\right| + C$

5. If f and g are twice differentiable and if $h(x) = f\left(g(x)\right)$, then $h''(x) =$

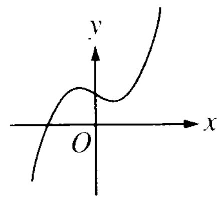
(A) $f''\left(g(x)\right)\left[g'(x)\right]^2 + f'\left(g(x)\right)g''(x)$

(B) $f''\left(g(x)\right)g'(x) + f'\left(g(x)\right)g''(x)$

(C) $f''\left(g(x)\right)\left[g'(x)\right]^2$

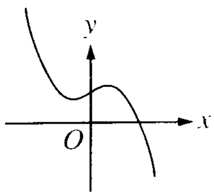
(D) $f''\left(g(x)\right)g''(x)$

(E) $f''\left(g(x)\right)$



6. The graph of $y = h(x)$ is shown above. Which of the following could be the graph of $y = h'(x)$?

(A)



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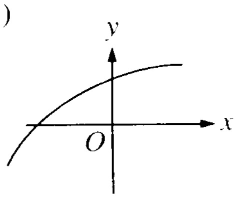

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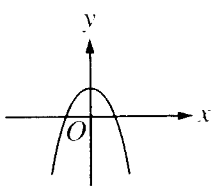

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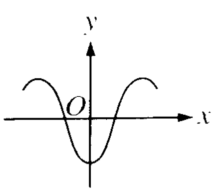
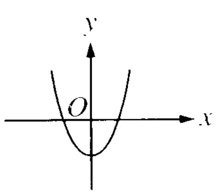


(E)



7. $\int_1^e \left(\frac{x^2-1}{x}\right) dx =$

(A) $e-\frac{1}{e}$ (B) e^2-e (C) $\frac{e^2}{2}-e+\frac{1}{2}$ (D) e^2-2 (E) $\frac{e^2}{2}-\frac{3}{2}$

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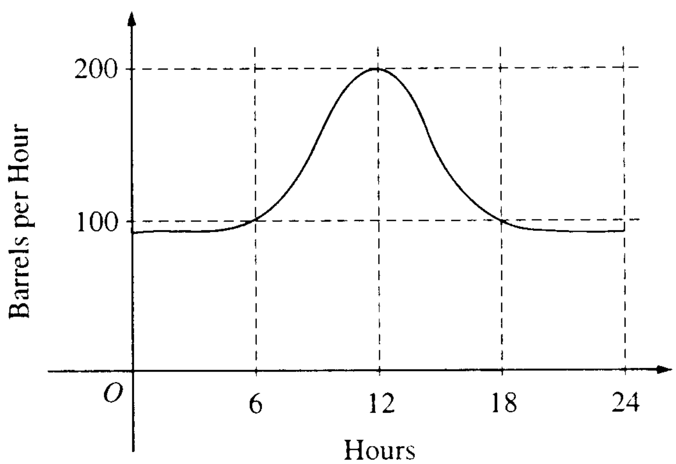
8. If $\frac{dy}{dx}=\sin x\cos^2 x$ and if $y=0$ when $x=\frac{\pi}{2}$, what is the value of y when $x=0$?

(A) -1 (B) $-\frac{1}{3}$ (C) 0 (D) $\frac{1}{3}$ (E) 1



9. The flow of oil, in barrels per hour, through a pipeline on July 9 is given by the graph shown above. Of the following, which best approximates the total number of barrels of oil that passed through the pipeline that day?

(A) 500 (B) 600 (C) 2,400 (D) 3,000 (E) 4,800

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10. A particle moves on a plane curve so that at any time $t>0$ its x-coordinate is t^3-t and its y-coordinate is $(2t-1)^3$. The acceleration vector of the particle at $t=1$ is

(A) $(0,1)$ (B) $(2,3)$ (C) $(2,6)$ (D) $(6,12)$ (E) $(6,24)$

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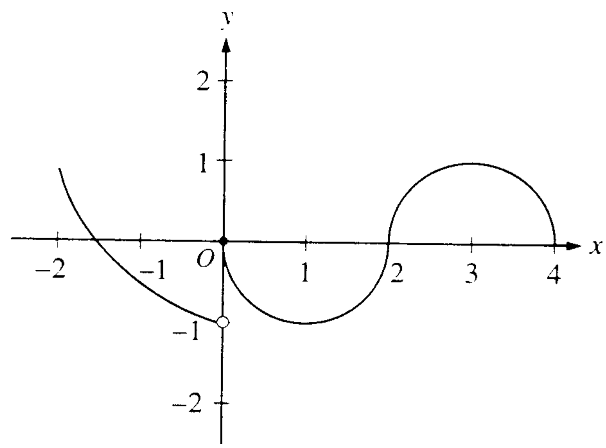
11. If f is a linear function and $0<a<b$, then $\int_a^b f''(x)\,dx =$

(A) 0 (B) 1 (C) $\frac{ab}{2}$ (D) $b-a$ (E) $\frac{b^2-a^2}{2}$

12. If $f(x)=\begin{cases}\ln x & \text{for } 0<x\le 2\\ x^2\ln 2 & \text{for } 2<x\le 4,\end{cases}$ then $\lim_{x\to 2} f(x)$ is

(A) $\ln 2$ (B) $\ln 8$ (C) $\ln 16$ (D) 4 (E) nonexistent



13. The graph of the function f shown in the figure above has a vertical tangent at the point $(2,0)$ and horizontal tangents at the points $(1,-1)$ and $(3,1)$. For what values of x, $-2<x<4$, is f not differentiable?

(A) 0 only (B) 0 and 2 only (C) 1 and 3 only (D) 0, 1, and 3 only (E) 0, 1, 2, and 3

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14. What is the approximation of the value of sin 1 obtained by using the fifth-degree Taylor polynomial about $x=0$ for $\sin x$?

(A) $1-\frac{1}{2}+\frac{1}{24}$

(B) $1-\frac{1}{2}+\frac{1}{4}$

(C) $1-\frac{1}{3}+\frac{1}{5}$

(D) $1-\frac{1}{4}+\frac{1}{8}$

(E) $1-\frac{1}{6}+\frac{1}{120}$

15. $\int x \cos x \, dx =$

(A) $x \sin x - \cos x + C$

(B) $x \sin x + \cos x + C$

(C) $-x \sin x + \cos x + C$

(D) $x \sin x + C$

(E) $\frac{1}{2}x^2 \sin x + C$

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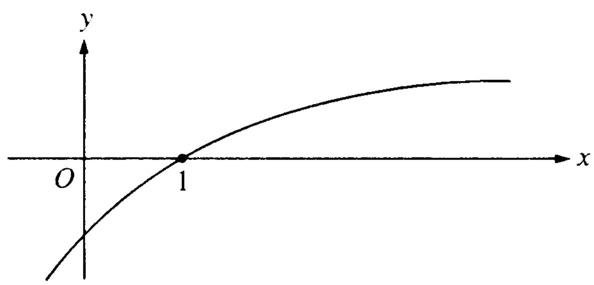
16. If f is the function defined by $f(x) = 3x^5 - 5x^4$, what are all the x-coordinates of points of inflection for the graph of f?

(A) -1 (B) 0 (C) 1 (D) 0 and 1 (E) -1, 0, and 1



17. The graph of a twice-differentiable function f is shown in the figure above. Which of the following is true?

(A) $f(1) < f'(1) < f''(1)$

(B) $f(1) < f''(1) < f'(1)$

(C) $f'(1) < f(1) < f''(1)$

(D) $f''(1) < f(1) < f'(1)$

(E) $f''(1) < f'(1) < f(1)$

18. Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{n}{n+2}$ II. $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$ III. $\sum_{n=1}^{\infty} \frac{1}{n}$

(A) None
(B) II only
(C) III only
(D) I and II only
(E) I and III only

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19. The area of the region inside the polar curve $r = 4\sin\theta$ and outside the polar curve $r = 2$ is given by

(A) $\frac{1}{2}\int_0^{\pi} (4\sin\theta - 2)^2 \, d\theta$ (B) $\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (4\sin\theta - 2)^2 \, d\theta$ (C) $\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (4\sin\theta - 2)^2 \, d\theta$

(D) $\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(16\sin^2\theta - 4\right) d\theta$ (E) $\frac{1}{2}\int_0^{\pi} \left(16\sin^2\theta - 4\right) d\theta$

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20. When $x = 8$, the rate at which $\sqrt[3]{x}$ is increasing is $\frac{1}{k}$ times the rate at which x is increasing. What is the value of k?

(A) 3 (B) 4 (C) 6 (D) 8 (E) 12

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21. The length of the path described by the parametric equations $x = \frac{1}{3}t^3$ and $y = \frac{1}{2}t^2$, where $0 \le t \le 1$, is given by

(A) $\int_0^1 \sqrt{t^2 + 1} \, dt$

(B) $\int_0^1 \sqrt{t^2 + t} \, dt$

(C) $\int_0^1 \sqrt{t^4 + t^2} \, dt$

(D) $\frac{1}{2}\int_0^1 \sqrt{4 + t^4} \, dt$

(E) $\frac{1}{6}\int_0^1 t^2 \sqrt{4t^2 + 9} \, dt$

22. If $\lim_{b\to\infty}\int_1^b \frac{dx}{x^p}$ is finite, then which of the following must be true?

(A) $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges

(B) $\sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges

(C) $\sum_{n=1}^{\infty}\frac{1}{n^{p-2}}$ converges

(D) $\sum_{n=1}^{\infty}\frac{1}{n^{p-1}}$ converges

(E) $\sum_{n=1}^{\infty}\frac{1}{n^{p+1}}$ diverges

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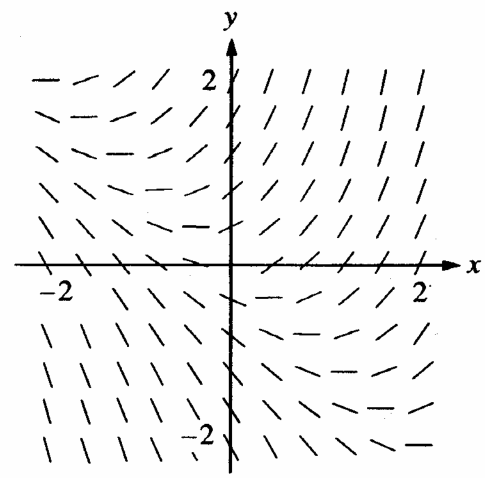
23. Let f be a function defined and continuous on the closed interval $[a,b]$. If f has a relative maximum at c and $a < c < b$, which of the following statements must be true?

I. $f'(c)$ exists.
II. If $f'(c)$ exists, then $f'(c) = 0$.
III. If $f''(c)$ exists, then $f''(c) \le 0$.

(A) II only (B) III only (C) I and II only (D) I and III only (E) II and III only



24. Shown above is a slope field for which of the following differential equations?

(A) $\frac{dy}{dx} = 1 + x$ (B) $\frac{dy}{dx} = x^2$ (C) $\frac{dy}{dx} = x + y$ (D) $\frac{dy}{dx} = \frac{x}{y}$ (E) $\frac{dy}{dx} = \ln y$

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25. $\int_0^{\infty} x^2 e^{-x^3}\,dx$ is

(A) $-\frac{1}{3}$ (B) 0 (C) $\frac{1}{3}$ (D) 1 (E) divergent

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26. The population $P(t)$ of a species satisfies the logistic differential equation $\frac{dP}{dt} = P\left(2 - \frac{P}{5000}\right)$, where the initial population $P(0) = 3{,}000$ and t is the time in years. What is $\lim_{t\to\infty} P(t)$?

(A) 2,500 (B) 3,000 (C) 4,200 (D) 5,000 (E) 10,000

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27. If $\sum_{n=0}^{\infty} a_n x^n$ is a Taylor series that converges to $f(x)$ for all real x, then $f'(1) =$

(A) 0 (B) a_1 (C) $\sum_{n=0}^{\infty} a_n$ (D) $\sum_{n=1}^{\infty} n a_n$ (E) $\sum_{n=1}^{\infty} n a_n^{\,n-1}$

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28. $\lim_{x\to 1} \frac{\int_1^x e^{t^2}\,dt}{x^2 - 1}$ is

(A) 0 (B) 1 (C) $\frac{e}{2}$ (D) e (E) nonexistent

1998 Calculus BC Solutions: Part A

1. C f will be increasing when its derivative is positive.
$f'(x)=3x^2+6x-9=3(x^2+2x-3)$ $f'(x)=3(x+3)(x-1)>0$ for $x<-3$ or $x>1$.

2. A $\frac{dx}{dt}=5$ and $\frac{dy}{dt}=3 \Rightarrow \frac{dy}{dx}=\frac{3}{5}$

3. D Find the derivative implicitly and substitute. $2y\cdot y'+3(xy+1)^2(x\cdot y'+y)=0$;
$2(-1)\cdot y'+3((2)(-1)+1)^2((2)\cdot y'+(-1))=0$; $-2y'+6\cdot y'-3=0$; $y'=\frac{3}{4}$

4. A Use partial fractions. $\frac{1}{x^2-6x+8}=\frac{1}{(x-4)(x-2)}=\frac{1}{2}\left(\frac{1}{x-4}-\frac{1}{x-2}\right)$

$$\int \frac{1}{x^2-6x+8}dx=\frac{1}{2}\left(\ln|x-4|-\ln|x-2|\right)+C=\frac{1}{2}\ln\left|\frac{x-4}{x-2}\right|+C$$

5. A $h'(x)=f'\left(g(x)\right)\cdot g'(x)$; $h''(x)=f''\left(g(x)\right)\cdot g'(x)\cdot g'(x)+f'\left(g(x)\right)\cdot g''(x)$
$h''(x)=f''\left(g(x)\right)\cdot\left(g'(x)\right)^2+f'\left(g(x)\right)\cdot g''(x)$

6. E The graph of h has 2 turning points and one point of inflection. The graph of h' will have 2 x-intercepts and one turning point. Only (C) and (E) are possible answers. Since the first turning point on the graph of h is a relative maximum, the first zero of h' must be a place where the sign changes from positive to negative. This is option (E).

7. E $\int_1^e \frac{x^2-1}{x}\,dx=\int_1^e x-\frac{1}{x}\,dx=\left(\frac{1}{2}x^2-\ln x\right)\Big|_1^e=\left(\frac{1}{2}e^2-1\right)-\left(\frac{1}{2}-0\right)=\frac{1}{2}e^2-\frac{3}{2}$

8. B $y(x)=-\frac{1}{3}\left(\cos x\right)^3+C$; Let $x=\frac{\pi}{2}$, $0=-\frac{1}{3}\left(\cos\frac{\pi}{2}\right)^3+C\Rightarrow C=0$. $y(0)=-\frac{1}{3}\left(\cos 0\right)^3=-\frac{1}{3}$

9. D Let $r(t)$ be the rate of oil flow as given by the graph, where t is measured in hours. The total number of barrels is given by $\int_0^{24} r(t)dt$. This can be approximated by counting the squares below the curve and above the horizontal axis. There are approximately five squares with area 600 barrels. Thus the total is about $3,000$ barrels.

10. E $v(t)=\left(3t^2-1,6(2t-1)^2\right)$ and $a(t)=\left(6t,24(2t-1)\right)\Rightarrow a(1)=(6,24)$

11. A Since f is linear, its second derivative is zero and the integral gives the area of a rectangle with zero height and width $(b-a)$. This area is zero.

12. E $\lim_{x\to 2^-} f(x) = \ln 2 \neq 4\ln 2 = \lim_{x\to 2^+} f(x)$. Therefore the limit does not exist.

13. B At $x = 0$ and $x = 2$ only. The graph has a non-vertical tangent line at every other point in the interval and so has a derivative at each of these other x's.

14. E $\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$; $\sin 1 \approx 1 - \frac{1^3}{3!} + \frac{1^5}{5!} = 1 - \frac{1}{6} + \frac{1}{120}$

15. B Use the technique of antiderivatives by parts. Let $u = x$ and $dv = \cos x\,dx$.

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$$

16. C Inflection point will occur when f'' changes sign. $f'(x) = 15x^4 - 20x^3$.
$f''(x) = 60x^3 - 60x^2 = 60x^2(x-1)$. The only sign change is at $x = 1$.

17. D From the graph $f(1) = 0$. Since $f'(1)$ represents the slope of the graph at $x = 1$, $f'(1) > 0$. Also, since $f''(1)$ represents the concavity of the graph at $x = 1$, $f''(1) < 0$.

18. B
I. Divergent. The limit of the nth term is not zero.
II. Convergent. This is the same as the alternating harmonic series.
III. Divergent. This is the harmonic series.

19. D Find the points of intersection of the two curves to determine the limits of integration.
$4\sin\theta = 2$ when $\sin\theta = 0.5$; this is at $\theta = \frac{\pi}{6}$ and $\frac{5\pi}{6}$. Area $= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left((4\sin\theta)^2 - (2)^2\right)d\theta$

20. E $\left.\frac{d\left(\sqrt[3]{x}\right)}{dt}\right|_{x=8} = \left.\frac{1}{3}x^{-\frac{2}{3}}\cdot\frac{dx}{dt}\right|_{x=8} = \frac{1}{3}(8)^{-\frac{2}{3}}\cdot\frac{dx}{dt} = \frac{1}{3}\cdot\frac{1}{4}\cdot\frac{dx}{dt} = \frac{1}{12}\cdot\frac{dx}{dt} \Rightarrow k = 12$

21 C The length of this parametric curve is given by $\int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^1 \sqrt{(t^2)^2 + t^2}\,dt$.

22. A This is the integral test applied to the series in (A). Thus the series in (A) converges. None of the others must be true.

23. E I. False. The relative maximum could be at a cusp.
II. True. There is a critical point at $x = c$ where $f'(c)$ exists
III. True. If $f''(c) > 0$, then there would be a relative minimum, not maximum

24. C All slopes along the diagonal $y = -x$ appear to be 0. This is consistent only with option (C). For each of the others you can see why they do not work. Option (A) does not work because all slopes at points with the same x coordinate would have to be equal. Option (B) does not work because all slopes would have to be positive. Option (D) does not work because all slopes in the third quadrant would have to be positive. Option (E) does not work because there would only be slopes for $y > 0$.

25. C $\int_0^\infty x^2 e^{-x^3}\,dx = \lim_{b\to\infty}\int_0^b x^2 e^{-x^3}\,dx = \lim_{b\to\infty} -\frac{1}{3}e^{-x^3}\Big|_0^b = \frac{1}{3}$.

26. E As $\lim_{t\to\infty}\frac{dP}{dt} = 0$ for a population satisfying a logistic differential equation, this means that $P \to 10{,}000$.

27. D If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then $f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1}$.

$f'(1) = \sum_{n=1}^{\infty} n a_n 1^{n-1} = \sum_{n=1}^{\infty} n a_n$

28. C Apply L'Hôpital's rule. $\lim_{x\to 1}\dfrac{\int_1^x e^{t^2}\,dt}{x^2 - 1} = \lim_{x\to 1}\dfrac{e^{x^2}}{2x} = \dfrac{e}{2}$