

*The AP Calculus  
Problem Book*



*Chuck Garner, Ph.D.*

**Dedicated to the students who used previous editions of this book!**

**BC Class of 2003**

Will Andersen  
Kenny Baskett  
Amanda Dugan  
Rochelle Dunlap  
Daniel Eisenman  
Kylene Farmer  
Nathan Garcia  
Brandon Jackson  
Drew Keenan  
Amin Makhani  
Patrick McGahee  
Rachel Meador  
Richard Moss  
Trent Phillips  
John Powell  
Blake Serra  
Jon Skypek  
David Thompson

**AB Class of 2003**

Ryan Boyd  
R. T. Collins  
Holly Ellington  
Stephen Gibbons  
Aly Mawji  
Chris McKnight  
Franklin Middlebrooks  
Kelly Morrison  
Julien Norton  
Jurod Russell

**BC Class of 2004**

Anita Amin  
Anushka Amin  
Rachel Atkinson  
Max Bernardy  
Lindsey Broadnax  
Andy Brunton  
Mitch Costley  
Caitlin Dingle  
Krista Firkus  
Justin Gilstrap  
Casey Haney  
Kendra Heisner  
Daniel Hendrix  
Candace Hogan  
Luke Hotchkiss  
Shawn Hyde  
Whitney Irwin  
Garett McLaurin  
C. K. Newman  
Matt Robuck  
Melissa Sanders  
Drew Sheffield  
Carsten Singh  
Andrea Smith  
Frankie Snavely  
Elizabeth Thai  
Ray Turner  
Timothy Van Heest  
Josh Williams  
Ben Wu  
Jodie Wu  
Michael Wysolowski  
Drew Yaun

**AB Class of 2004**

Brooke Atkinson  
Kevin Dirth  
Jaimy Lee  
Robert Parr  
Garion Reddick  
Andre Russell  
Megan Villanueva

**BC Class of 2005**

Jonathan Andersen  
James Bascle  
Eryn Bernardy  
Joseph Bost  
William Brawley  
Alex Hamilton  
Sue Ann Hollowell  
Vicky Johnson  
Kayla Koch  
Amy Lanchester  
Dana McKnight  
Kathryn Moore  
Ryan Moore  
Candace Murphy  
Hannah Newman  
Bre' Ana Paige  
Lacy Reynolds  
Jacob Schiefer  
Kevin Todd  
Amanda Wallace  
Jonathan Wysolowski  
Keeli Zanders

**AB Class of 2005**

Lacey Avery  
Alicia Bellis  
Mollie Bogle  
Carin Godemann  
Shawn Kumar  
Joe Madsen  
Julie Matthews  
Jazmine Reaves  
Sarah Singh  
Andrew Vanstone  
Jawaan Washington  
Michael Westbury  
Jeremy Wilkerson

**BC Class of 2006**

John Barnett  
Chelsea Britt  
Sam Brotherton  
Justin Carlin  
Ryan Ceciliani  
Nicole Fraute  
Carin Godemann  
Megan Harris  
Zack Higbie  
Nayoon Kim  
Amy Kovac  
Julie Leber  
Salman Makhani  
Julie Matthews  
Chris Meador  
Sophia Newton  
Chris Randall  
Nicole Richardson  
Britt Schneider  
Michaela Simoes  
Jesse Smith

**AB Class of 2006**

Shail Amin  
Lucky Baker  
Savannah Barrus  
Taylor Boggus  
Justin Clemons  
Kim Dang  
Will Gibson  
Latiria Hill  
John-Lee Hughey  
Michael Hyman  
Cassandra Lohmeyer  
Chris Long  
Jason Long  
Joseph Long  
Cason Lowe  
Cassie Lowury  
Victoria Nesmith  
Dion Roseberry  
Cassie Smith  
Aniya Watson

**BC Class of 2007**

Betsey Avery  
Aaron Bullock  
Cecily Bullock  
Daniel Chen  
Raymond Clunie  
Jim Creager  
Katie Dugan  
Mitchell Granade  
Allyse Keel  
Jacob Kovac  
Jan Lauritsen  
Ally Long  
Chris Long  
Nick Macie  
Will Martin  
Justin McKithen  
Ruhy Momin  
Steven Rouk  
Tyler Sigwald  
Lauren Troxler  
Ryan Young

**AB Class of 2007**

Melanie Allen  
Briana Brimidge  
Michelle Dang  
Kyle Davis  
Holly Dean  
Shaunna Duggan  
Chris Elder  
Kevin Gorman  
Jessie Holmes  
Gary McCrear  
Faith Middlebrooks  
D'Andra Myers  
Brandi Paige  
Miriam Perfecto  
Thomas Polstra  
Torri Preston  
Heather Quinn  
Andrew Smith  
Nicole Thomas

**BC Class of 2008**

Layla Bouzoubaa  
Kevin Brawley  
Ashley Chackalayil  
Nate Coursey  
Michelle Dang  
Kathryn Daniel  
Justin Easley  
Samantha Girardot  
Jessie Holmes  
Johnathan Johnson  
Kathryn Johnson  
Tyler Kelly  
Rochelle Lobo  
Monica Longoria  
April Lovering  
Kevin Masters  
David McCalley  
Gary McCrear  
Patti Murphy  
Sarah Pace  
Bijal Patel  
Kunal Patel  
Miriam Perfecto  
Thomas Polstra  
James Rives  
Khaliliah Smith  
Andrew Stover  
Aswad Walker  
Ashley Williams

**AB Class of 2008**

Omair Akhtar  
John Collins  
Rachel Delevett  
Kelsey Hinely  
Sarah Kustick  
Bianca Manahu  
Shonette McCalmon  
Ansley Mitcham  
Tesia Olgetree  
Lauren Powell  
Xan Reynolds  
Lauren Stewart  
Matt Wann  
Ryan Young

# *The AP Calculus Problem Book*



*Chuck Garner, Ph.D.*

*Rockdale Magnet School  
for Science and Technology*

*Fourth Edition,  
Revised and Corrected, 2008*

## *The AP Calculus Problem Book*

### Publication history:

*First edition, 2002*

*Second edition, 2003*

*Third edition, 2004*

*Third edition Revised and Corrected, 2005*

*Fourth edition, 2006, Edited by Amy Lanchester*

*Fourth edition Revised and Corrected, 2007*

*Fourth edition, Corrected, 2008*

This book was produced directly from the author's  $\text{\LaTeX}$  files.

Figures were drawn by the author using the  $\text{\TeXdraw}$  package.

TI-Calculator screen-shots produced by a TI-83Plus calculator using a TI-Graph Link.

$\text{\LaTeX}$  (pronounced "Lay-Tek") is a document typesetting program (not a word processor) that is available free from [www.miktex.org](http://www.miktex.org), which also includes  $\text{\TeXnicCenter}$ , a free and easy-to-use user-interface.

---

# Contents

---

<b>1</b>	<b>LIMITS</b>	<b>7</b>
1.1	Graphs of Functions . . . . .	8
1.2	The Slippery Slope of Lines . . . . .	9
1.3	The Power of Algebra . . . . .	10
1.4	Functions Behaving Badly . . . . .	11
1.5	Take It to the Limit . . . . .	12
1.6	One-Sided Limits . . . . .	13
1.7	One-Sided Limits (Again) . . . . .	14
1.8	Limits Determined by Graphs . . . . .	15
1.9	Limits Determined by Tables . . . . .	16
1.10	The Possibilities Are Limitless... . . . .	17
1.11	Average Rates of Change: Episode I . . . . .	18
1.12	Exponential and Logarithmic Functions . . . . .	18
1.13	Average Rates of Change: Episode II . . . . .	19
1.14	Take It To the Limit—One More Time . . . . .	20
1.15	Solving Equations . . . . .	21
1.16	Continuously Considering Continuity . . . . .	22
1.17	Have You Reached the Limit? . . . . .	23
1.18	Multiple Choice Questions on Limits . . . . .	24
1.19	Sample A.P. Problems on Limits . . . . .	26
	Last Year’s Limits Test . . . . .	27
<b>2</b>	<b>DERIVATIVES</b>	<b>35</b>
2.1	Negative and Fractional Exponents . . . . .	36
2.2	Logically Thinking About Logic . . . . .	37
2.3	The Derivative By Definition . . . . .	38
2.4	Going Off on a Tangent . . . . .	39
2.5	Six Derivative Problems . . . . .	40
2.6	Trigonometry: a Refresher . . . . .	41

2.7	Continuity and Differentiability . . . . .	42
2.8	The RULES: Power Product Quotient Chain . . . . .	43
2.9	Trigonometric Derivatives . . . . .	44
2.10	Tangents, Normals, and Continuity (Revisited) . . . . .	45
2.11	Implicit Differentiation . . . . .	46
2.12	The Return of Geometry . . . . .	47
2.13	Meet the Rates (They're Related) . . . . .	48
2.14	Rates Related to the Previous Page . . . . .	49
2.15	Excitement with Derivatives! . . . . .	50
2.16	Derivatives of Inverses . . . . .	51
2.17	Dérivé, Derivado, Ableitung, Derivative . . . . .	52
2.18	Sample A.P. Problems on Derivatives . . . . .	54
2.19	Multiple-Choice Problems on Derivatives . . . . .	56
	Last Year's Derivatives Test . . . . .	58
<b>3</b>	<b>APPLICATIONS of DERIVATIVES</b>	<b>67</b>
3.1	The Extreme Value Theorem . . . . .	68
3.2	Rolle to the Extreme with the Mean Value Theorem . . . . .	69
3.3	The First and Second Derivative Tests . . . . .	70
3.4	Derivatives and Their Graphs . . . . .	71
3.5	Two Derivative Problems . . . . .	73
3.6	Sketching Functions . . . . .	74
3.7	Problems of Motion . . . . .	76
3.8	Maximize or Minimize? . . . . .	78
3.9	More Tangents and Derivatives . . . . .	80
3.10	More Excitement with Derivatives! . . . . .	81
3.11	Bodies, Particles, Rockets, Trucks, and Canals . . . . .	82
3.12	Even More Excitement with Derivatives! . . . . .	84
3.13	Sample A.P. Problems on Applications of Derivatives . . . . .	86
3.14	Multiple-Choice Problems on Applications of Derivatives . . . . .	89
	Last Year's Applications of Derivatives Test . . . . .	92
<b>4</b>	<b>INTEGRALS</b>	<b>101</b>
4.1	The ANTIderivative! . . . . .	102
4.2	Derivative Rules Backwards . . . . .	103
4.3	The Method of Substitution . . . . .	104
4.4	Using Geometry for Definite Integrals . . . . .	105
4.5	Some Riemann Sums . . . . .	106
4.6	The MVT and the FTC . . . . .	107
4.7	The FTC, Graphically . . . . .	108
4.8	Definite and Indefinite Integrals . . . . .	109
4.9	Integrals Involving Logarithms and Exponentials . . . . .	110
4.10	It Wouldn't Be Called the Fundamental Theorem If It Wasn't Fundamental . . . . .	111
4.11	Definite and Indefinite Integrals Part 2 . . . . .	113
4.12	Regarding Riemann Sums . . . . .	114
4.13	Definitely Exciting Definite Integrals! . . . . .	116
4.14	How Do I Find the Area Under Thy Curve? Let Me Count the Ways... . . . .	117
4.15	Three Integral Problems . . . . .	118

4.16	Trapezoid and Simpson . . . . .	119
4.17	Properties of Integrals . . . . .	120
4.18	Sample A.P. Problems on Integrals . . . . .	121
4.19	Multiple Choice Problems on Integrals . . . . .	124
	Last Year's Integrals Test . . . . .	127
<b>5</b>	<b>APPLICATIONS of INTEGRALS</b>	<b>135</b>
5.1	Volumes of Solids with Defined Cross-Sections . . . . .	136
5.2	Turn Up the Volume! . . . . .	137
5.3	Volume and Arc Length . . . . .	138
5.4	Differential Equations, Part One . . . . .	139
5.5	The Logistic Curve . . . . .	140
5.6	Differential Equations, Part Two . . . . .	141
5.7	Slope Fields and Euler's Method . . . . .	142
5.8	Differential Equations, Part Three . . . . .	143
5.9	Sample A.P. Problems on Applications of Integrals . . . . .	144
5.10	Multiple Choice Problems on Application of Integrals . . . . .	147
	Last Year's Applications of Integrals Test . . . . .	150
<b>6</b>	<b>TECHNIQUES of INTEGRATION</b>	<b>159</b>
6.1	A Part, And Yet, Apart... . . . . .	160
6.2	Partial Fractions . . . . .	161
6.3	Trigonometric Substitution . . . . .	162
6.4	Four Integral Problems . . . . .	163
6.5	L'Hôpital's Rule . . . . .	164
6.6	Improper Integrals! . . . . .	165
6.7	The Art of Integration . . . . .	166
6.8	Functions Defined By Integrals . . . . .	168
6.9	Sample A.P. Problems on Techniques of Integration . . . . .	170
6.10	Sample Multiple-Choice Problems on Techniques of Integration . . . . .	173
	Last Year's Techniques of Integration Test . . . . .	175
<b>7</b>	<b>SERIES, VECTORS, PARAMETRICS and POLAR</b>	<b>183</b>
7.1	Sequences: Bounded and Unbounded . . . . .	184
7.2	It is a Question of Convergence... . . . .	185
7.3	Infinite Sums . . . . .	186
7.4	Tests for Convergence and Divergence . . . . .	187
7.5	More Questions of Convergence... . . . .	188
7.6	Power Series! . . . . .	189
7.7	Maclaurin Series . . . . .	190
7.8	Taylor Series . . . . .	191
7.9	Vector Basics . . . . .	192
7.10	Calculus with Vectors and Parametrics . . . . .	193
7.11	Vector-Valued Functions . . . . .	194
7.12	Motion Problems with Vectors . . . . .	195
7.13	Polar Basics . . . . .	196
7.14	Differentiation (Slope) and Integration (Area) in Polar . . . . .	197
7.15	Sample A.P. Problems on Series, Vectors, Parametrics, and Polar . . . . .	198

7.16	Sample Multiple-Choice Problems on Series, Vectors, Parametrics, and Polar	201
	Last Year's Series, Vectors, Parametrics, and Polar Test	203
<b>8</b>	<b>AFTER THE A.P. EXAM</b>	<b>211</b>
8.1	Hyperbolic Functions	212
8.2	Surface Area of a Solid of Revolution	213
8.3	Linear First Order Differential Equations	214
8.4	Curvature	215
8.5	Newton's Method	216
<b>9</b>	<b>PRACTICE and REVIEW</b>	<b>217</b>
9.1	Algebra	218
9.2	Derivative Skills	219
9.3	Can You Stand All These Exciting Derivatives?	220
9.4	Different Differentiation Problems	222
9.5	Integrals... Again!	224
9.6	Intégrale, Integrale, Integraal, Integral	225
9.7	Calculus Is an Integral Part of Your Life	226
9.8	Particles	227
9.9	Areas	228
9.10	The Deadly Dozen	229
9.11	Two Volumes and Two Differential Equations	230
9.12	Differential Equations, Part Four	231
9.13	More Integrals	232
9.14	Definite Integrals Requiring Definite Thought	233
9.15	Just When You Thought Your Problems Were Over...	234
9.16	Interesting Integral Problems	236
9.17	Infinitely Interesting Infinite Series	238
9.18	Getting Serious About Series	239
9.19	A Series of Series Problems	240
<b>10</b>	<b>GROUP INVESTIGATIONS</b>	<b>241</b>
	About the Group Investigations	242
10.1	Finding the Most Economical Speed for Trucks	243
10.2	Minimizing the Area Between a Graph and Its Tangent	243
10.3	The Ice Cream Cone Problem	243
10.4	Designer Polynomials	244
10.5	Inventory Management	244
10.6	Optimal Design of a Steel Drum	246
<b>11</b>	<b>CALCULUS LABS</b>	<b>247</b>
	About the Labs	248
1:	The Intermediate Value Theorem	250
2:	Local Linearity	252
3:	Exponentials	254
4:	A Function and Its Derivative	256
5:	Riemann Sums and Integrals	259
6:	Numerical Integration	262



---

7: Indeterminate Limits and l'Hôpital's Rule . . . . .	267
8: Sequences . . . . .	270
9: Approximating Functions by Polynomials . . . . .	272
10: Newton's Method . . . . .	274
<b>12 TI-CALCULATOR LABS</b>	<b>277</b>
Before You Start . . . . .	278
1: Useful Stuff . . . . .	279
2: Derivatives . . . . .	281
3: Maxima, Minima, Inflections . . . . .	283
4: Integrals . . . . .	284
5: Approximating Integrals . . . . .	286
6: Approximating Integrals II . . . . .	287
7: Applications of Integrals . . . . .	289
8: Differential Equations . . . . .	292
9: Sequences and Series . . . . .	293
<b>13 CHALLENGE PROBLEMS</b>	<b>295</b>
Set A . . . . .	296
Set B . . . . .	297
Set C . . . . .	299
Set D . . . . .	301
Set E . . . . .	303
Set F . . . . .	305
<b>A FORMULAS</b>	<b>309</b>
Formulas from Geometry . . . . .	310
Greek Alphabet . . . . .	311
Trigonometry . . . . .	312
<b>B SUCCESS IN MATHEMATICS</b>	<b>315</b>
Calculus BC Syllabus . . . . .	316
<b>C ANSWERS</b>	<b>329</b>
Answers to Last Year's Tests . . . . .	343



# CHAPTER 1

---

## LIMITS

---

## 1.1 Graphs of Functions

DESCRIBE THE GRAPHS OF EACH OF THE FOLLOWING FUNCTIONS USING ONLY ONE OF THE FOLLOWING TERMS: *line, parabola, cubic, hyperbola, semicircle*.

1.  $y = x^3 + 5x^2 - x - 1$

7.  $y = \frac{-3}{x-5}$

2.  $y = \frac{1}{x}$

8.  $y = 9 - x^2$

3.  $y = 3x + 2$

9.  $y = -3x^3$

4.  $y = -x^3 + 500x$

10.  $y = 34x - 5^2$

5.  $y = \sqrt{9 - x^2}$

11.  $y = 34x^2 - 52$

6.  $y = x^2 + 4$

12.  $y = \sqrt{1 - x^2}$

GRAPH THE FOLLOWING FUNCTIONS ON YOUR CALCULATOR ON THE WINDOW  $-3 \leq x \leq 3$ ,  $-2 \leq y \leq 2$ . SKETCH WHAT YOU SEE. CHOOSE ONE OF THE FOLLOWING TO DESCRIBE WHAT HAPPENS TO THE GRAPH AT THE ORIGIN: A) GOES VERTICAL; B) FORMS A CUSP; C) GOES HORIZONTAL; OR D) STOPS AT ZERO.

13.  $y = x^{1/3}$

17.  $y = x^{1/4}$

14.  $y = x^{2/3}$

18.  $y = x^{5/4}$

15.  $y = x^{4/3}$

19.  $y = x^{1/5}$

16.  $y = x^{5/3}$

20.  $y = x^{2/5}$

21. Based on the answers from the problems above, find a pattern for the behavior of functions with exponents of the following forms:  $x^{\text{even/odd}}$ ,  $x^{\text{odd/odd}}$ ,  $x^{\text{odd/even}}$ .

GRAPH THE FOLLOWING FUNCTIONS ON YOUR CALCULATOR IN THE STANDARD WINDOW AND SKETCH WHAT YOU SEE. AT WHAT VALUE(S) OF  $x$  ARE THE FUNCTIONS EQUAL TO ZERO?

22.  $y = |x - 1|$

25.  $y = |4 + x^2|$

23.  $y = |x^2 - 4|$

26.  $y = |x^3| - 8$

24.  $y = |x^3 - 8|$

27.  $y = |x^2 - 4x - 5|$

---

In the company of friends, writers can discuss their books, economists the state of the economy, lawyers their latest cases, and businessmen their latest acquisitions, but mathematicians cannot discuss their mathematics at all. And the more profound their work, the less understandable it is. —*Alfred Adler*

## 1.2 The Slippery Slope of Lines

The point-slope form of a line is

$$m(x - x_1) = y - y_1.$$

IN THE FIRST SIX PROBLEMS, FIND THE EQUATION OF THE LINE WITH THE GIVEN PROPERTIES.

28. slope:  $\frac{2}{3}$ ; passes through  $(2, 1)$
29. slope:  $-\frac{1}{4}$ ; passes through  $(0, 6)$
30. passes through  $(3, 6)$  and  $(2, 7)$
31. passes through  $(-6, 1)$  and  $(1, 1)$
32. passes through  $(5, -4)$  and  $(5, 9)$
33. passes through  $(10, 3)$  and  $(-10, 3)$
34. A line passes through  $(1, 2)$  and  $(2, 5)$ . Another line passes through  $(0, 0)$  and  $(-4, 3)$ . Find the point where the two lines intersect.
35. A line with slope  $-\frac{2}{5}$  and passing through  $(2, 4)$  is parallel to another line passing through  $(-3, 6)$ . Find the equations of both lines.
36. A line with slope  $-3$  and passing through  $(1, 5)$  is perpendicular to another line passing through  $(1, 1)$ . Find the equations of both lines.
37. A line passes through  $(8, 8)$  and  $(-2, 3)$ . Another line passes through  $(3, -1)$  and  $(-3, 0)$ . Find the point where the two lines intersect.
38. The function  $f(x)$  is a line. If  $f(3) = 5$  and  $f(4) = 9$ , then find the equation of the line  $f(x)$ .
39. The function  $f(x)$  is a line. If  $f(0) = 4$  and  $f(12) = 5$ , then find the equation of the line  $f(x)$ .
40. The function  $f(x)$  is a line. If the slope of  $f(x)$  is 3 and  $f(2) = 5$ , then find  $f(7)$ .
41. The function  $f(x)$  is a line. If the slope of  $f(x)$  is  $\frac{2}{3}$  and  $f(1) = 1$ , then find  $f(\frac{3}{2})$ .
42. If  $f(2) = 1$  and  $f(b) = 4$ , then find the value of  $b$  so that the line  $f(x)$  has slope 2.
43. Find the equation of the line that has  $x$ -intercept at 4 and  $y$ -intercept at 1.
44. Find the equation of the line with slope 3 which intersects the semicircle  $y = \sqrt{25 - x^2}$  at  $x = 4$ .

### 1.3 The Power of Algebra

FACTOR EACH OF THE FOLLOWING COMPLETELY.

45.  $y^2 - 18y + 56$

52.  $x^3 + 8$

46.  $33u^2 - 37u + 10$

53.  $8x^2 + 27$

47.  $c^2 + 9c - 8$

54.  $64x^6 - 1$

48.  $(x - 6)^2 - 9$

55.  $(x + 2)^3 + 125$

49.  $3(x + 9)^2 - 36(x + 9) + 81$

56.  $x^3 - 2x^2 + 9x - 18$

50.  $63q^3 - 28q$

57.  $p^5 - 5p^3 + 8p^2 - 40$

51.  $2\pi r^2 + 2\pi r + hr + h$

SIMPLIFY EACH OF THE FOLLOWING EXPRESSIONS.

58.  $\frac{3(x - 4) + 2(x + 5)}{6(x - 4)}$

61.  $\frac{\frac{9x^2}{3}}{\frac{5x^3}{x}}$

59.  $\frac{1}{x - y} - \frac{1}{y - x}$

62.  $\frac{y}{1 - \frac{1}{y}}$

60.  $3x - \frac{5x - 7}{4}$

63.  $\frac{x}{1 - \frac{1}{y}} + \frac{y}{1 - \frac{1}{x}}$

RATIONALIZE EACH OF THE FOLLOWING EXPRESSIONS.

64.  $\frac{-3 + 9\sqrt{7}}{\sqrt{7}}$

67.  $\frac{2 - \sqrt{3}}{4 + \sqrt{3}}$

70.  $\frac{5x}{\sqrt{x + 5} - \sqrt{5}}$

65.  $\frac{3\sqrt{2} + \sqrt{5}}{2\sqrt{10}}$

68.  $\frac{x - 6}{\sqrt{x - 3} + \sqrt{3}}$

71.  $\frac{2\sqrt{5} - 6\sqrt{3}}{4\sqrt{5} + \sqrt{3}}$

66.  $\frac{2x + 8}{\sqrt{x + 4}}$

69.  $\frac{9}{\sqrt{2x + 3} - \sqrt{2x}}$

72.  $\frac{x}{\sqrt{x + 3} - \sqrt{3}}$

---

Incubation is the work of the subconscious during the waiting time, which may be several years. Illumination, which can happen in a fraction of a second, is the emergence of the creative idea into the conscious. This almost always occurs when the mind is in a state of relaxation, and engaged lightly with ordinary matters. Illumination implies some mysterious rapport between the subconscious and the conscious, otherwise emergence would not happen. What rings the bell at the right moment? —*John E. Littlewood*

## 1.4 Functions Behaving Badly

SKETCH A GRAPH OF EACH FUNCTION, THEN FIND ITS DOMAIN.

$$73. G(x) = \begin{cases} x^2 & x \geq -1 \\ 2x + 3 & x < -1 \end{cases}$$

$$76. V(r) = \begin{cases} \sqrt{1-r^2} & -1 \leq r \leq 1 \\ \frac{1}{r} & r > 1 \end{cases}$$

$$74. A(t) = \begin{cases} |t| & t < 1 \\ -3t + 4 & t \geq 1 \end{cases}$$

$$77. U(x) = \begin{cases} 1/x & x < -1 \\ x & -1 \leq x \leq 1 \\ 1/x & x > 1 \end{cases}$$

$$75. h(x) = x + |x|$$

$$78. f(x) = \frac{x}{|x|}$$

FOR THE FOLLOWING, FIND A) THE DOMAIN; B) THE  $y$ -INTERCEPT; AND C) ALL VERTICAL AND HORIZONTAL ASYMPTOTES.

$$79. y = \frac{x-2}{x}$$

$$82. y = \frac{x}{x^2 + 2x - 8}$$

$$80. y = \frac{-1}{(x-1)^2}$$

$$83. y = \frac{x^2 - 2x}{x^2 - 16}$$

$$81. y = \frac{x-2}{x-3}$$

$$84. y = \frac{x^2 - 4x + 3}{x-4}$$

CHOOSE THE BEST ANSWER.

85. Which of the following represents the graph of  $f(x)$  moved to the left 3 units?

- A)  $f(x-3)$                       B)  $f(x)-3$                       C)  $f(x+3)$                       D)  $f(x)+3$

86. Which of the following represents the graph of  $g(x)$  moved to the right 2 units and down 7 units?

- A)  $g(x-2)-7$                       B)  $g(x+2)+7$                       C)  $g(x+7)-2$                       D)  $g(x-7)+2$

FACTOR EACH OF THE FOLLOWING.

$$87. 49p^2 - 144q^2$$

$$90. 8x^3 - 27$$

$$88. 15z^2 + 52z + 32$$

$$91. 27x^3 + y^3$$

$$89. x^3 - 8$$

$$92. 2w^3 - 10w^2 + w - 5$$

---

He gets up in the morning and immediately starts to do calculus. And in the evening he plays his bongo drums. —Mrs. Feynman's reasons cited for divorcing her husband, Richard Feynman, Nobel prize-winning physicist

## 1.5 Take It to the Limit

EVALUATE EACH LIMIT.

93.  $\lim_{x \rightarrow -2} (3x^2 - 2x + 1)$

94.  $\lim_{x \rightarrow 5} 4$

95.  $\lim_{x \rightarrow -3} (x^3 - 2)$

96.  $\lim_{z \rightarrow 8} \frac{z^2 - 64}{z - 8}$

97.  $\lim_{t \rightarrow 1/4} \frac{4t - 1}{1 - 16t^2}$

98.  $\lim_{x \rightarrow -2} \frac{x^2 + 5x + 6}{x^2 - 4}$

99.  $\lim_{x \rightarrow 1/3} \frac{3x^2 - 7x + 2}{-6x^2 + 5x - 1}$

100.  $\lim_{p \rightarrow 4} \frac{p^3 - 64}{4 - p}$

101.  $\lim_{k \rightarrow -1} \sqrt[3]{\frac{3k - 5}{25k - 2}}$

102.  $\lim_{x \rightarrow 2} \sqrt{\frac{x^2 - 4}{2x^2 + x - 6}}$

103.  $\lim_{x \rightarrow 0} \frac{x}{\sqrt{x + 3} - \sqrt{3}}$

104.  $\lim_{y \rightarrow 0} \frac{\sqrt{3y + 2} - \sqrt{2}}{y}$

105. Let  $F(x) = \frac{3x - 1}{9x^2 - 1}$ . Find  $\lim_{x \rightarrow 1/3} F(x)$ . Is this the same as the value of  $F(\frac{1}{3})$ ?

106. Let  $G(x) = \frac{4x^2 - 3x}{4x - 3}$ . Find  $\lim_{x \rightarrow 3/4} G(x)$ . Is this the same as the value of  $G(\frac{3}{4})$ ?

107. Let  $P(x) = \begin{cases} 3x - 2 & x \neq \frac{1}{3} \\ 4 & x = \frac{1}{3} \end{cases}$ . Find  $\lim_{x \rightarrow 1/3} P(x)$ . Is this the same as the value of  $P(\frac{1}{3})$ ?

108. Let  $Q(x) = \begin{cases} \frac{x^2 - 16}{x - 4} & x \neq 4 \\ 3 & x = 4 \end{cases}$ . Find  $\lim_{x \rightarrow 4} Q(x)$ . Is this the same as the value of  $Q(4)$ ?

SOLVE EACH SYSTEM OF EQUATIONS.

109. 
$$\begin{cases} 2x - 3y = -4 \\ 5x + y = 7 \end{cases}$$

110. 
$$\begin{cases} 6x + 15y = 8 \\ 3x - 20y = -7 \end{cases}$$

111. If  $F(x) = \begin{cases} 2x - 5 & x > \frac{1}{2} \\ 3kx - 1 & x < \frac{1}{2} \end{cases}$  then find the value of  $k$  such that  $\lim_{x \rightarrow 1/2} F(x)$  exists.



## 1.6 One-Sided Limits

FIND THE LIMITS, IF THEY EXIST, AND FIND THE INDICATED VALUE. IF A LIMIT DOES NOT EXIST, EXPLAIN WHY.

112. Let  $f(x) = \begin{cases} 4x - 2 & x > 1 \\ 2 - 4x & x \leq 1. \end{cases}$

a)  $\lim_{x \rightarrow 1^+} f(x)$                       b)  $\lim_{x \rightarrow 1^-} f(x)$                       c)  $\lim_{x \rightarrow 1} f(x)$                       d)  $f(1)$

113. Let  $a(x) = \begin{cases} 3 - 6x & x > 1 \\ -1 & x = 1 \\ x^2 & x < 1. \end{cases}$

a)  $\lim_{x \rightarrow 1^+} a(x)$                       b)  $\lim_{x \rightarrow 1^-} a(x)$                       c)  $\lim_{x \rightarrow 1} a(x)$                       d)  $a(1)$

114. Let  $h(t) = \begin{cases} 3t - 1 & t > 2 \\ -5 & t = 2 \\ 1 + 2t & t < 2. \end{cases}$

a)  $\lim_{t \rightarrow 2^+} h(t)$                       b)  $\lim_{t \rightarrow 2^-} h(t)$                       c)  $\lim_{t \rightarrow 2} h(t)$                       d)  $h(2)$

115. Let  $c(x) = \begin{cases} x^2 - 9 & x < 3 \\ 5 & x = 3 \\ 9 - x^2 & x > 3. \end{cases}$

a)  $\lim_{x \rightarrow 3^+} c(x)$                       b)  $\lim_{x \rightarrow 3^-} c(x)$                       c)  $\lim_{x \rightarrow 3} c(x)$                       d)  $c(3)$

116. Let  $v(t) = |3t - 6|$ .

a)  $\lim_{t \rightarrow 2^+} v(t)$                       b)  $\lim_{t \rightarrow 2^-} v(t)$                       c)  $\lim_{t \rightarrow 2} v(t)$                       d)  $v(2)$

117. Let  $y(x) = \frac{|3x|}{x}$ .

a)  $\lim_{x \rightarrow 0^+} y(x)$                       b)  $\lim_{x \rightarrow 0^-} y(x)$                       c)  $\lim_{x \rightarrow 0} y(x)$                       d)  $y(0)$

118. Let  $k(z) = |-2z + 4| - 3$ .

a)  $\lim_{z \rightarrow 2^+} k(z)$                       b)  $\lim_{z \rightarrow 2^-} k(z)$                       c)  $\lim_{z \rightarrow 2} k(z)$                       d)  $k(2)$

EXPLAIN WHY THE FOLLOWING LIMITS DO NOT EXIST.

119.  $\lim_{x \rightarrow 0} \frac{x}{|x|}$

120.  $\lim_{x \rightarrow 1} \frac{1}{x - 1}$

## 1.7 One-Sided Limits (Again)

IN THE FIRST NINE PROBLEMS, EVALUATE EACH LIMIT.

$$121. \lim_{x \rightarrow 5^+} \frac{x-5}{x^2-25}$$

$$124. \lim_{x \rightarrow 4^-} \frac{3x}{16-x^2}$$

$$127. \lim_{x \rightarrow 2^-} \frac{x+2}{2-x}$$

$$122. \lim_{x \rightarrow 2^+} \frac{2-x}{x^2-4}$$

$$125. \lim_{x \rightarrow 0} \frac{x^2-7}{3x^3-2x}$$

$$128. \lim_{x \rightarrow 4^+} \frac{3x}{x^2-4}$$

$$123. \lim_{x \rightarrow 2} \frac{|x-2|}{x-2}$$

$$126. \lim_{x \rightarrow 0^-} \left( \frac{3}{x^2} - \frac{2}{x} \right)$$

$$129. \lim_{x \rightarrow 0} \frac{x^2}{\sqrt{3x^2+1}-1}$$

SOLVE EACH SYSTEM OF EQUATIONS.

$$130. \begin{cases} x-y = -7 \\ \frac{1}{2}x + 3y = 14 \end{cases}$$

$$131. \begin{cases} 8x-5y = 1 \\ 5x-8y = -1 \end{cases}$$

$$132. \text{ If } G(x) = \begin{cases} 3x^2 - kx + m & x \geq 1 \\ mx - 2k & -1 < x < 1 \\ -3m + 4x^3k & x \leq -1 \end{cases} \text{ then find the values of } m \text{ and } k \text{ such that both } \lim_{x \rightarrow 1} G(x) \text{ and } \lim_{x \rightarrow -1} G(x) \text{ exist.}$$

FOR THE FOLLOWING, FIND A) THE DOMAIN; B) THE  $y$ -INTERCEPT; AND C) ALL VERTICAL AND HORIZONTAL ASYMPTOTES.

$$133. y = \frac{x^3 + 3x^2}{x^4 - 4x^2}$$

$$134. y = \frac{x^5 - 25x^3}{x^4 + 2x^3}$$

$$135. y = \frac{x^2 + 6x + 9}{2x}$$

SUPPOSE THAT  $\lim_{x \rightarrow 4} f(x) = 5$  AND  $\lim_{x \rightarrow 4} g(x) = -2$ . FIND THE FOLLOWING LIMITS.

$$136. \lim_{x \rightarrow 4} f(x)g(x)$$

$$139. \lim_{x \rightarrow 4} xf(x)$$

$$137. \lim_{x \rightarrow 4} (f(x) + 3g(x))$$

$$140. \lim_{x \rightarrow 4} (g(x))^2$$

$$138. \lim_{x \rightarrow 4} \frac{f(x)}{f(x) - g(x)}$$

$$141. \lim_{x \rightarrow 4} \frac{g(x)}{f(x) - 1}$$

---

How can you shorten the subject? That stern struggle with the multiplication table, for many people not yet ended in victory, how can you make it less? Square root, as obdurate as a hardwood stump in a pasture, nothing but years of effort can extract it. You can't hurry the process. Or pass from arithmetic to algebra; you can't shoulder your way past quadratic equations or ripple through the binomial theorem. Instead, the other way; your feet are impeded in the tangled growth, your pace slackens, you sink and fall somewhere near the binomial theorem with the calculus in sight on the horizon. So died, for each of us, still bravely fighting, our mathematical training; except for a set of people called "mathematicians" – born so, like crooks. —*Stephen Leacock*

## 1.8 Limits Determined by Graphs

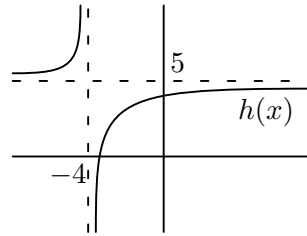
REFER TO THE GRAPH OF  $h(x)$  TO EVALUATE THE FOLLOWING LIMITS.

142.  $\lim_{x \rightarrow -4^+} h(x)$

143.  $\lim_{x \rightarrow -4^-} h(x)$

144.  $\lim_{x \rightarrow \infty} h(x)$

145.  $\lim_{x \rightarrow -\infty} h(x)$



REFER TO THE GRAPH OF  $g(x)$  TO EVALUATE THE FOLLOWING LIMITS.

146.  $\lim_{x \rightarrow a^+} g(x)$

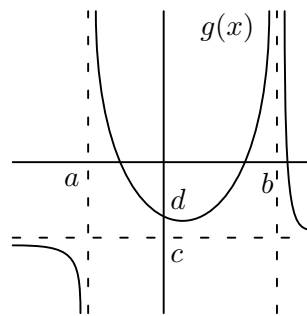
151.  $\lim_{x \rightarrow b^-} g(x)$

147.  $\lim_{x \rightarrow a^-} g(x)$

148.  $\lim_{x \rightarrow 0} g(x)$

149.  $\lim_{x \rightarrow \infty} g(x)$

150.  $\lim_{x \rightarrow b^+} g(x)$



REFER TO THE GRAPH OF  $f(x)$  TO DETERMINE WHICH STATEMENTS ARE TRUE AND WHICH ARE FALSE. IF A STATEMENT IS FALSE, EXPLAIN WHY.

152.  $\lim_{x \rightarrow -1^+} f(x) = 1$

159.  $\lim_{x \rightarrow 1} f(x) = 1$

153.  $\lim_{x \rightarrow 0^-} f(x) = 0$

160.  $\lim_{x \rightarrow 1} f(x) = 0$

154.  $\lim_{x \rightarrow 0^-} f(x) = 1$

161.  $\lim_{x \rightarrow 2^-} f(x) = 2$

155.  $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x)$

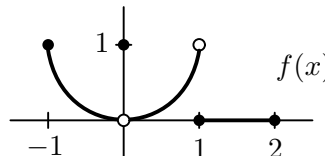
162.  $\lim_{x \rightarrow -1^-} f(x)$  does not exist

156.  $\lim_{x \rightarrow 0} f(x)$  exists

163.  $\lim_{x \rightarrow 2^+} f(x) = 0$

157.  $\lim_{x \rightarrow 0} f(x) = 0$

158.  $\lim_{x \rightarrow 0} f(x) = 1$



## 1.9 Limits Determined by Tables

USING YOUR CALCULATOR, FILL IN EACH OF THE FOLLOWING TABLES TO FIVE DECIMAL PLACES. USING THE INFORMATION FROM THE TABLE, DETERMINE EACH LIMIT. (FOR THE TRIGONOMETRIC FUNCTIONS, YOUR CALCULATOR MUST BE IN *radian* MODE.)

164.  $\lim_{x \rightarrow 0} \frac{\sqrt{x+3} - \sqrt{3}}{x}$

$x$		-0.1	-0.01	-0.001	0.001	0.01	0.1
$\frac{\sqrt{x+3} - \sqrt{3}}{x}$							

165.  $\lim_{x \rightarrow -3} \frac{\sqrt{1-x} - 2}{x+3}$

$x$		-3.1	-3.01	-3.001	-2.999	-2.99	-2.9
$\frac{\sqrt{1-x} - 2}{x+3}$							

166.  $\lim_{x \rightarrow 0} \frac{\sin x}{x}$

$x$		-0.1	-0.01	-0.001	0.001	0.01	0.1
$\frac{\sin x}{x}$							

167.  $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x}$

$x$		-0.1	-0.01	-0.001	0.001	0.01	0.1
$\frac{1 - \cos x}{x}$							

168.  $\lim_{x \rightarrow 0} (1+x)^{1/x}$

$x$		-0.1	-0.01	-0.001	0.001	0.01	0.1
$(1+x)^{1/x}$							

169.  $\lim_{x \rightarrow 1} x^{1/(1-x)}$

$x$		0.9	0.99	0.999	1.001	1.01	1.1
$x^{1/(1-x)}$							

---

Science is built up with facts, as a house is with stones. But a collection of facts is no more a science than a heap of stones is a house. —*Henri Poincaré*

## 1.10 The Possibilities Are Limitless...

REFER TO THE GRAPH OF  $R(x)$  TO EVALUATE THE FOLLOWING.

170.  $\lim_{x \rightarrow \infty} R(x)$

178.  $\lim_{x \rightarrow b} R(x)$

171.  $\lim_{x \rightarrow -\infty} R(x)$

179.  $\lim_{x \rightarrow c} R(x)$

172.  $\lim_{x \rightarrow a^+} R(x)$

180.  $\lim_{x \rightarrow d} R(x)$

173.  $\lim_{x \rightarrow a^-} R(x)$

181.  $\lim_{x \rightarrow e} R(x)$

174.  $\lim_{x \rightarrow a} R(x)$

182.  $R(e)$

175.  $\lim_{x \rightarrow 0} R(x)$

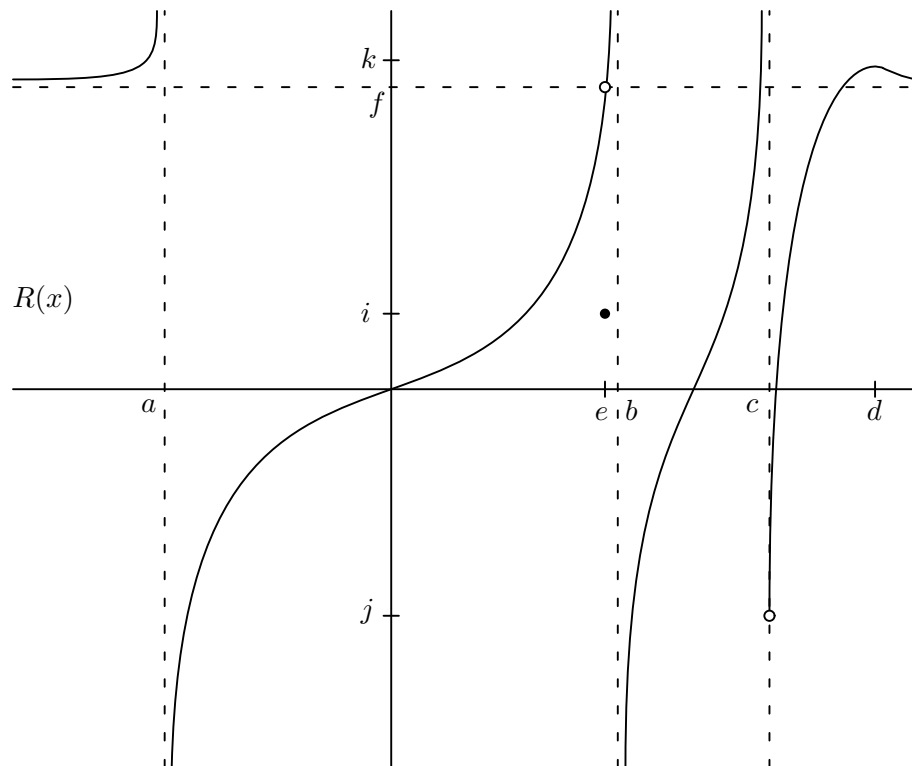
183.  $R(0)$

176.  $\lim_{x \rightarrow b^+} R(x)$

184.  $R(b)$

177.  $\lim_{x \rightarrow b^-} R(x)$

185.  $R(d)$




---

One of the big misapprehensions about mathematics that we perpetrate in our classrooms is that the teacher always seems to know the answer to any problem that is discussed. This gives students the idea that there is a book somewhere with all the right answers to all of the interesting questions, and that teachers know those answers. And if one could get hold of the book, one would have everything settled. That's so unlike the true nature of mathematics. —Leon Hankin

## 1.11 Average Rates of Change: Episode I

**186.** Find a formula for the average rate of change of the area of a circle as its radius  $r$  changes from 3 to some number  $x$ . Then determine the average rate of change of the area of a circle as the radius  $r$  changes from

- a) 3 to 3.5                      b) 3 to 3.2                      c) 3 to 3.1                      d) 3 to 3.01

**187.** Find a formula for the average rate of change of the volume of a cube as its side length  $s$  changes from 2 to some number  $x$ . Then determine the average rate of change of the volume of a cube as the side length  $s$  changes from

- a) 2 to 3                      b) 2 to 2.5                      c) 2 to 2.2                      d) 2 to 2.1

**188.** A car is stopped at a traffic light and begins to move forward along a straight road when the light turns green. The distance  $s$ , in feet, traveled by a car in  $t$  seconds is given by  $s(t) = 2t^2$  ( $0 \leq t \leq 30$ ). What is the average rate of change of the car from

- a)  $t = 0$  to  $t = 5$               b)  $t = 5$  to  $t = 10$               c)  $t = 0$  to  $t = 10$               d)  $t = 10$  to  $t = 10.1$

IN THE FOLLOWING SIX PROBLEMS, FIND A FORMULA FOR THE AVERAGE RATE OF CHANGE OF EACH FUNCTION FROM  $x = 1$  TO SOME NUMBER  $x = c$ .

**189.**  $f(x) = x^2 + 2x$

**192.**  $g(t) = 2t - 6$

**190.**  $f(x) = \sqrt{x}$

**193.**  $p(x) = \frac{3}{x}$

**191.**  $f(x) = 2x^2 - 4x$

**194.**  $F(x) = -2x^3$

## 1.12 Exponential and Logarithmic Functions

SIMPLIFY THE FOLLOWING EXPRESSIONS.

**195.**  $e^{\ln x + \ln y}$

**197.**  $\log_4(4^{y+3})$

**199.**  $\ln(e^{5x + \ln 6})$

**196.**  $\ln(e^{3x})$

**198.**  $5^{\log_5(x+2y)}$

**200.**  $e^{3 \ln x - 2 \ln 5}$

FOR THE FOLLOWING FUNCTIONS, FIND THE DOMAIN AND THE  $y$ -INTERCEPT.

**201.**  $y = e^{3x-1} \sqrt{x}$

**204.**  $y = \ln(8x^2 - 4)$

**202.**  $y = x \log_3(5x - 2)$

**205.**  $y = e^{5x/(3x-2)} \ln e^x$

**203.**  $y = e^{3x/(2x-1)} \sqrt[3]{x-7}$

**206.**  $y = \ln(x^2 - 8x + 15)$

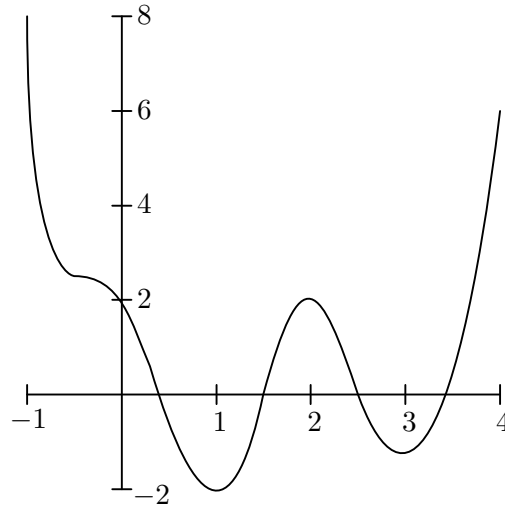
---

Ludwig Boltzmann, who spent much of his life studying statistical mechanics, died in 1906, by his own hand. Paul Ehrenfest, carrying on the work, died similarly in 1933. Now it is our turn to study statistical mechanics. —David L. Goodstein, in the preface to his book *States of Matter*

### 1.13 Average Rates of Change: Episode II

**207.** The position  $p(t)$  is given by the graph at the right.

- Find the average velocity of the object between times  $t = 1$  and  $t = 4$ .
- Find the equation of the secant line of  $p(t)$  between times  $t = 1$  and  $t = 4$ .
- For what times  $t$  is the object's velocity positive? For what times is it negative?



**208.** Suppose  $f(1) = 2$  and the average rate of change of  $f$  between 1 and 5 is 3. Find  $f(5)$ .

**209.** The position  $p(t)$ , in meters, of an object at time  $t$ , in seconds, along a line is given by  $p(t) = 3t^2 + 1$ .

- Find the change in position between times  $t = 1$  and  $t = 3$ .
- Find the average velocity of the object between times  $t = 1$  and  $t = 4$ .
- Find the average velocity of the object between any time  $t$  and another time  $t + \Delta t$ .

**210.** Let  $f(x) = x^2 + x - 2$ .

- Find the average rate of change of  $f(x)$  between times  $x = -1$  and  $x = 2$ .
- Draw the graph of  $f$  and the graph of the secant line through  $(-1, -2)$  and  $(2, 4)$ .
- Find the slope of the secant line graphed in part b) and then find an equation of this secant line.
- Find the average rate of change of  $f(x)$  between any point  $x$  and another point  $x + \Delta x$ .

FIND THE AVERAGE RATE OF CHANGE OF EACH FUNCTION OVER THE GIVEN INTERVALS.

**211.**  $f(x) = x^3 + 1$  over a)  $[2, 3]$ ; b)  $[-1, 1]$

**213.**  $h(t) = \frac{1}{\tan t}$  over a)  $[\frac{\pi}{4}, \frac{3\pi}{4}]$ ; b)  $[\frac{\pi}{6}, \frac{\pi}{3}]$

**212.**  $R(x) = \sqrt{4x+1}$  over a)  $[0, \frac{3}{4}]$ ; b)  $[0, 2]$

**214.**  $g(t) = 2 + \cos t$  over a)  $[0, \pi]$ ; b)  $[-\pi, \pi]$

## 1.14 Take It To the Limit—One More Time

EVALUATE EACH LIMIT.

$$215. \lim_{x \rightarrow \infty} \frac{5x - 3}{3 - 2x}$$

$$216. \lim_{y \rightarrow \infty} \frac{4y - 3}{3 - 2y}$$

$$217. \lim_{x \rightarrow \infty} \frac{3x^2 + 2x + 1}{5 - 2x^2 + 3x}$$

$$218. \lim_{x \rightarrow \infty} \frac{3x + 2}{4x^2 - 3}$$

$$219. \lim_{x \rightarrow \infty} \frac{4x^2 - 3}{3x + 2}$$

$$220. \lim_{x \rightarrow \infty} \frac{3x^3 - 1}{4x + 3}$$

$$221. \lim_{x \rightarrow \infty} \left( 4x + \frac{3}{x^2} \right)$$

$$222. \lim_{z \rightarrow \infty} \frac{\sqrt{z^2 + 9}}{z + 9}$$

$$223. \lim_{x \rightarrow \infty} \frac{3}{x^5}$$

$$224. \lim_{x \rightarrow -2} \frac{5x - 1}{x + 2}$$

$$225. \lim_{x \rightarrow 5} \frac{-4x + 3}{x - 5}$$

$$226. \lim_{x \rightarrow 0} \left( 3 - \frac{2}{x} \right)$$

$$227. \lim_{x \rightarrow 0} \left( 3 - \frac{2}{x^2} \right)$$

$$228. \lim_{x \rightarrow 5} \frac{3x^2}{x^2 - 25}$$

$$229. \lim_{x \rightarrow 0} \frac{\sqrt{x+3} - \sqrt{3}}{x}$$

$$230. \lim_{x \rightarrow -3} \frac{x^2 - 5x + 6}{x^2 - 9}$$

$$231. \lim_{x \rightarrow -3} (3x + 2)$$

$$232. \lim_{x \rightarrow 2} (-x^2 + x - 2)$$

$$233. \lim_{x \rightarrow 4} \sqrt[3]{x+4}$$

$$234. \lim_{x \rightarrow 2} \frac{1}{x}$$

$$235. \lim_{x \rightarrow 3} \frac{\sqrt{x+1}}{x-4}$$

$$236. \lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - 1}$$

$$237. \lim_{x \rightarrow 0} \frac{\sqrt{2+x} - \sqrt{2}}{x}$$

$$238. \lim_{x \rightarrow \infty} \frac{\sqrt{2+x} - \sqrt{2}}{x}$$

FOR THE FOLLOWING, A) SKETCH THE GRAPH OF  $f$  AND B) DETERMINE AT WHAT POINTS  $c$  IN THE DOMAIN OF  $f$ , IF ANY, DOES  $\lim_{x \rightarrow c} f(x)$  EXIST. JUSTIFY YOUR ANSWER.

$$239. f(x) = \begin{cases} 3 - x & x < 2 \\ \frac{x}{2} + 1 & x > 2 \end{cases}$$

$$240. f(x) = \begin{cases} 3 - x & x < 2 \\ 2 & x = 2 \\ \frac{x}{2} & x > 2 \end{cases}$$

$$241. f(x) = \begin{cases} \frac{1}{x-1} & x < 1 \\ x^3 - 2x + 5 & x \geq 1 \end{cases}$$

$$242. f(x) = \begin{cases} 1 - x^2 & x \neq -1 \\ 2 & x = -1 \end{cases}$$

$$243. f(x) = \begin{cases} \sqrt{1-x^2} & 0 \leq x < 1 \\ 1 & 1 \leq x < 2 \\ 2 & x = 2 \end{cases}$$

$$244. f(x) = \begin{cases} x & -1 \leq x < 0 \text{ or } 0 < x \leq 1 \\ 1 & x = 0 \\ 0 & x < -1 \text{ or } x > 1 \end{cases}$$

---

The discovery in 1846 of the planet Neptune was a dramatic and spectacular achievement of mathematical astronomy. The very existence of this new member of the solar system, and its exact location, were demonstrated with pencil and paper; there was left to observers only the routine task of pointing their telescopes at the spot the mathematicians had marked. —James R. Newman



## 1.15 Solving Equations

SOLVE EACH OF THE FOLLOWING EQUATIONS.

245.  $1 - \frac{8}{k^3} = 0$

246.  $4p^3 - 4p = 0$

247.  $x^3 - 2x^2 - 3x = 0$

248.  $3x^2 - 10x - 8 = 0$

249.  $|4x^3 - 3| = 0$

250.  $|w^2 - 6w| = 9$

251.  $\frac{3(x-4) - (3x-2)}{(x-4)^2} = 0$

252.  $\frac{2x-3}{2(x^2-3x)} = 0$

253.  $2 \ln x = 9$

254.  $e^{5x} = 7$

255.  $\ln(2x - 1) = 0$

256.  $e^{3x+7} = 12$

257.  $\ln \sqrt[4]{x+1} = \frac{1}{2}$

258.  $2^{3x-1} = \frac{1}{2}$

259.  $\log_8(x-5) = \frac{2}{3}$

260.  $\log \sqrt{z} = \log(z-6)$

261.  $2 \ln(p+3) - \ln(p+1) = 3 \ln 2$

262.  $3^{x^2} = 7$

263.  $\log_3(3x) = \log_3 x + \log_3(4-x)$

FIND ALL REAL ZEROS OF THE FOLLOWING FUNCTIONS.

264.  $y = x^2 - 4$

265.  $y = -2x^4 + 5$

266.  $y = x^3 - 3$

267.  $y = x^3 - 9x$

268.  $y = x^4 + 2x^2$

269.  $y = x^3 - 4x^2 - 5x$

270.  $y = x^3 - 5x^2 - x + 5$

271.  $y = x^3 + 3x^2 - 4x - 12$

272.  $y = \frac{x-2}{x}$

273.  $y = \frac{-1}{(x-1)^2}$

274.  $y = \frac{1+x}{1-x}$

275.  $y = \frac{x^3}{1+x^2}$

276.  $y = \frac{x^2-2x}{x^2-16}$

277.  $y = \frac{x^2-4x+3}{x-4}$

278.  $y = \frac{x^3+3x^2}{x^4-4x^2}$

279.  $y = \frac{x^5 - 25x^3}{x^4 + 2x^3}$

280.  $y = x^2 + \frac{1}{x}$

281.  $y = e^{3x-1} \sqrt{x}$

282.  $y = x \log_3(5x-2)$

283.  $y = e^{3x/(2x-1)} \sqrt[3]{x-7}$

284.  $y = \ln(8x^2 - 4)$

285.  $y = e^{5x/(3x-2)} \ln e^x$

DETERMINE WHETHER THE FUNCTIONS IN THE PROBLEMS LISTED ARE EVEN, ODD, OR NEITHER.

286. problem 264

288. problem 272

290. problem 275

287. problem 268

289. problem 274

291. problem 280

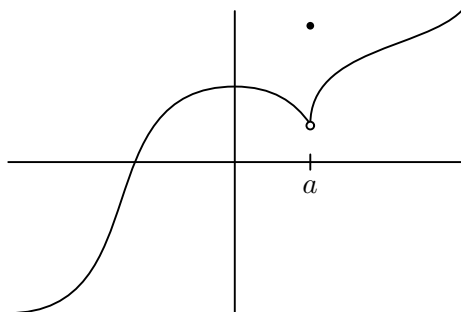
---

The chief aim of all investigations of the external world should be to discover the rational order and harmony which has been imposed on it by God and which He revealed to us in the language of mathematics. —*Johannes Kepler*

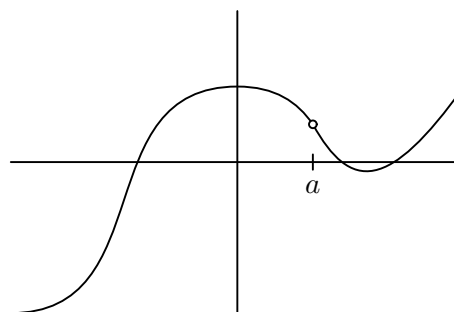
## 1.16 Continuously Considering Continuity

EXAMINE THE GRAPHS OF THE FUNCTIONS BELOW. EXPLAIN WHY EACH IS DISCONTINUOUS AT  $x = a$ , AND DETERMINE THE TYPE OF DISCONTINUITY.

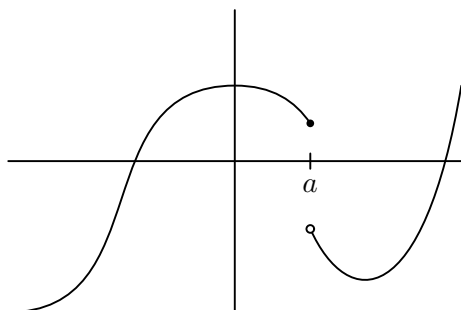
292.



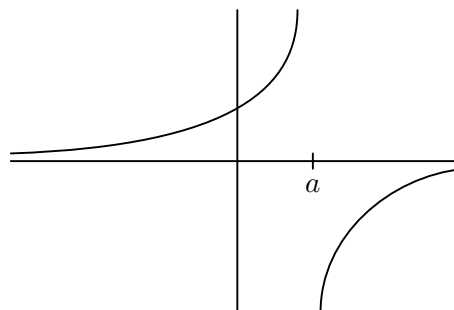
294.



293.



295.



DETERMINE THE VALUES OF THE INDEPENDENT VARIABLE FOR WHICH THE FUNCTION IS DISCONTINUOUS. JUSTIFY YOUR ANSWERS.

$$296. f(x) = \frac{x^2 + x - 2}{x - 1}$$

$$297. d(r) = \frac{r^4 - 1}{r^2 - 1}$$

$$298. A(k) = \frac{k^2 - 2}{k^4 - 1}$$

$$299. q(t) = \frac{3}{t + 7}$$

$$300. m(z) = \begin{cases} \frac{z^2 + z - 2}{z - 1} & z \neq 1 \\ 3 & z = 1 \end{cases}$$

$$301. s(w) = \begin{cases} \frac{3}{w + 7} & w \neq -7 \\ 2 & w = -7 \end{cases}$$

$$302. p(j) = \begin{cases} 4 & j < 0 \\ 0 & j = 0 \\ \sqrt{j} & j > 0 \end{cases}$$

$$303. b(y) = \begin{cases} y^2 - 9 & y < 3 \\ 5 & y = 3 \\ 9 - y^2 & y > 3 \end{cases}$$

---

Considering how many fools can calculate, it is surprising that it should be thought either a difficult or tedious task for any other fool to learn to master the same tricks. —*Silvanus P. Thompson*

### 1.17 Have You Reached the Limit?

**304.** Estimate the value of  $\lim_{x \rightarrow \infty} (\sqrt{x^2 + x + 1} - x)$  by graphing or by making a table of values.

**305.** Estimate the value of  $\lim_{x \rightarrow \infty} (\sqrt{x^2 + x} - \sqrt{x^2 - x})$  by graphing or by making a table of values.

**306.** Consider the function  $f(x) = \begin{cases} x^2 - 1 & -1 \leq x < 0 \\ 2x & 0 < x < 1 \\ 1 & x = 1 \\ -2x + 4 & 1 < x < 2 \\ 0 & 2 < x < 3. \end{cases}$

a) Graph this function.

i) Is  $f$  continuous at  $x = 1$ ?

b) Does  $f(-1)$  exist?

j) Is  $f$  defined at  $x = 2$ ?

c) Does  $\lim_{x \rightarrow -1^+} f(x)$  exist?

k) Is  $f$  continuous at  $x = 2$ ?

d) Does  $\lim_{x \rightarrow -1^+} f(x) = f(-1)$ ?

l) At what values of  $x$  is  $f$  continuous?

e) Is  $f$  continuous at  $x = -1$ ?

m) What value should be assigned to  $f(2)$  to make the function continuous at  $x = 2$ ?

f) Does  $f(1)$  exist?

g) Does  $\lim_{x \rightarrow 1^+} f(x)$  exist?

n) To what new value of  $f(1)$  be changed to remove the discontinuity?

h) Does  $\lim_{x \rightarrow 1^+} f(x) = f(1)$ ?

**307.** Is  $F(x) = \frac{|x^2 - 4|x}{x + 2}$  continuous everywhere? Why or why not?

**308.** Is  $F(x) = \frac{|x^2 + 4x|(x + 2)}{x + 4}$  continuous everywhere? Why or why not?

FIND THE CONSTANTS  $a$  AND  $b$  SUCH THAT THE FUNCTION IS CONTINUOUS EVERYWHERE.

**309.**  $f(x) = \begin{cases} x^3 & x \leq 2 \\ ax^2 & x > 2 \end{cases}$

**310.**  $g(x) = \begin{cases} \frac{4 \sin x}{x} & x < 0 \\ a - 2x & x \geq 0 \end{cases}$

**311.**  $f(x) = \begin{cases} 2 & x \leq -1 \\ ax + b & -1 < x < 3 \\ -2 & x \geq 3 \end{cases}$

**312.**  $g(x) = \begin{cases} \frac{x^2 - a^2}{x - a} & x \neq a \\ 8 & x = a \end{cases}$

**1.18 Multiple Choice Questions on Limits**

313.  $\lim_{x \rightarrow \infty} \frac{3x^4 - 2x + 1}{7x - 8x^5 - 1} =$   
A)  $\infty$                       B)  $-\infty$                       C) 0                      D)  $\frac{3}{7}$                       E)  $-\frac{3}{8}$
314.  $\lim_{x \rightarrow 0^-} \frac{1}{x} =$   
A)  $\infty$                       B)  $-\infty$                       C) 0                      D) 1                      E) does not exist
315.  $\lim_{x \rightarrow 1/3} \frac{9x^2 - 1}{3x - 1} =$   
A)  $\infty$                       B)  $-\infty$                       C) 0                      D) 2                      E) 3
316.  $\lim_{x \rightarrow 0} \frac{x^3 - 8}{x^2 - 4} =$   
A) 4                      B) 0                      C) 1                      D) 3                      E) 2
317. In order for the line  $y = a$  to be a horizontal asymptote of  $h(x)$ , which of the following must be true?  
A)  $\lim_{x \rightarrow a^+} h(x) = \infty$   
B)  $\lim_{x \rightarrow a^-} h(x) = -\infty$   
C)  $\lim_{x \rightarrow \infty} h(x) = \infty$   
D)  $\lim_{x \rightarrow -\infty} h(x) = a$   
E)  $\lim_{x \rightarrow -\infty} h(x) = \infty$
318. The function  $G(x) = \begin{cases} x - 3 & x > 2 \\ -5 & x = 2 \\ 3x - 7 & x < 2 \end{cases}$  is not continuous at  $x = 2$  because  
A)  $G(2)$  is not defined  
B)  $\lim_{x \rightarrow 2} G(x)$  does not exist  
C)  $\lim_{x \rightarrow 2} G(x) \neq G(2)$   
D)  $G(2) \neq -5$   
E) All of the above
319.  $\lim_{x \rightarrow 0} \frac{3x^2 + 2x}{2x + 1} =$   
A)  $\infty$                       B)  $-\infty$                       C) 0                      D) 1                      E)  $\frac{3}{2}$

320.  $\lim_{x \rightarrow -1/2^-} \frac{2x^2 - 3x - 2}{2x + 1} =$   
 A)  $\infty$                       B)  $-\infty$                       C) 1                      D)  $\frac{3}{2}$                       E)  $-\frac{5}{2}$
321.  $\lim_{x \rightarrow -2} \frac{\sqrt{2x + 5} - 1}{x + 2} =$   
 A) 1                      B) 0                      C)  $\infty$                       D)  $-\infty$                       E) does not exist
322.  $\lim_{x \rightarrow -\infty} \frac{3x^2 + 2x^3 + 5}{x^4 + 7x^2 - 3} =$   
 A) 0                      B) 2                      C)  $\frac{3}{7}$                       D)  $\infty$                       E)  $-\infty$
323.  $\lim_{x \rightarrow 0} \frac{-x^2 + 4}{x^2 - 1} =$   
 A) 1                      B) 0                      C)  $-4$                       D)  $-1$                       E)  $\infty$
324. The function  $G(x) = \begin{cases} x^2 & x > 2 \\ 4 - 2x & x < 2 \end{cases}$  is not continuous at  $x = 2$  because  
 A)  $G(2)$  does not exist  
 B)  $\lim_{x \rightarrow 2} G(x)$  does not exist  
 C)  $\lim_{x \rightarrow 2} G(x) = G(2)$   
 D) All three statements A, B, and C  
 E) None of the above
325. The domain of the function  $f(x) = \sqrt{4 - x^2}$  is  
 A)  $x < -2$  or  $x > 2$     B)  $x \leq -2$  or  $x \geq 2$     C)  $-2 < x < 2$     D)  $-2 \leq x \leq 2$     E)  $x \leq 2$
326.  $\lim_{x \rightarrow 5} \frac{x^2 - 25}{x - 5} =$   
 A) 0                      B) 10                      C)  $-10$                       D) 5                      E) does not exist
327. Find  $k$  so that  $f(x) = \begin{cases} \frac{x^2 - 16}{x - 4} & x \neq 4 \\ k & x = 4 \end{cases}$  is continuous for all  $x$ .  
 A) any value                      B) 0                      C) 8                      D) 16                      E) no value

### 1.19 Sample A.P. Problems on Limits

**328.** For the function  $f(x) = \frac{2x - 1}{|x|}$ , find the following:

- a)  $\lim_{x \rightarrow \infty} f(x)$ ;
- b)  $\lim_{x \rightarrow -\infty} f(x)$ ;
- c)  $\lim_{x \rightarrow 0^+} f(x)$ ;
- d)  $\lim_{x \rightarrow 0^-} f(x)$ ;
- e) All horizontal asymptotes;
- f) All vertical asymptotes.

**329.** Consider the function  $h(x) = \frac{1}{1 - 2^{1/x}}$ .

- a) What is the domain of  $h$ ?
- b) Find all zeros of  $h$ .
- c) Find all vertical and horizontal asymptotes of  $h$ .
- d) Find  $\lim_{x \rightarrow 0^+} h(x)$ .
- e) Find  $\lim_{x \rightarrow 0^-} h(x)$ .
- f) Find  $\lim_{x \rightarrow 0} h(x)$ .

**330.** Consider the function  $g(x) = \frac{\sin |x|}{x}$  defined for all real numbers.

- a) Is  $g(x)$  an even function, an odd function, or neither? Justify your answer.
- b) Find the zeros and the domain of  $g$ .
- c) Find  $\lim_{x \rightarrow 0} g(x)$ .

**331.** Let  $f(x) = \begin{cases} \sqrt{1 - x^2} & 0 \leq x < 1 \\ 1 & 1 \leq x < 2. \\ 2 & x = 2 \end{cases}$ .

- a) Draw the graph of  $f$ .
- b) At what points  $c$  in the domain of  $f$  does  $\lim_{x \rightarrow c} f(x)$  exist?
- c) At what points does only the left-hand limit exist?
- d) At what points does only the right-hand limit exist?

**A.P. Calculus Test One**  
**Section One**  
**Multiple-Choice**  
**No Calculators**  
Time—30 minutes  
Number of Questions—15

The scoring for this section is determined by the formula

$$[C - (0.25 \times I)] \times 1.8$$

where  $C$  is the number of correct responses and  $I$  is the number of incorrect responses. An unanswered question earns zero points. The maximum possible points earned on this section is 27, which represents 50% of the total test score.

*Directions:* Solve each of the following problems, using the available space for scratch work. After examining the form of the choices, decide which is the best of the choices given and fill in the corresponding choice on your answer sheet. Do not spend too much time on any one problem.

Good Luck!

NAME:

1. Which of the following is continuous at  $x = 0$  ?

- I.  $f(x) = |x|$
- II.  $f(x) = e^x$
- III.  $f(x) = \ln(e^x - 1)$

- A) I only
  - B) II only
  - C) I and II only
  - D) II and III only
  - E) none of these
- 

2. The graph of a function  $f$  is reflected across the  $x$ -axis and then shifted up 2 units. Which of the following describes this transformation on  $f$  ?

- A)  $-f(x)$
  - B)  $f(x) + 2$
  - C)  $-f(x + 2)$
  - D)  $-f(x - 2)$
  - E)  $-f(x) + 2$
- 

3. Which of the following functions is *not* continuous for all real numbers  $x$  ?

- A)  $f(x) = x^{1/3}$
- B)  $f(x) = \frac{2}{(x + 1)^4}$
- C)  $f(x) = |x + 1|$
- D)  $f(x) = \sqrt{1 + e^x}$
- E)  $f(x) = \frac{x - 3}{x^2 + 9}$



4.  $\lim_{x \rightarrow 1} \frac{\ln x}{x}$  is

- A) 1
  - B) 0
  - C)  $e$
  - D)  $-e$
  - E) nonexistent
- 

5.  $\lim_{x \rightarrow 0} \left( \frac{1}{x} + \frac{1}{x^2} \right) =$

- A) 0
  - B)  $\frac{1}{2}$
  - C) 1
  - D) 2
  - E)  $\infty$
- 

6.  $\lim_{x \rightarrow \infty} \frac{x^3 - 4x + 1}{2x^3 - 5} =$

- A)  $-\frac{1}{5}$
- B)  $\frac{1}{2}$
- C)  $\frac{2}{3}$
- D) 1
- E) Does not exist

7. For what value of  $k$  does  $\lim_{x \rightarrow 4} \frac{x^2 - x + k}{x - 4}$  exist?

- A)  $-12$
  - B)  $-4$
  - C)  $3$
  - D)  $7$
  - E) No such value exists.
- 

8.  $\lim_{x \rightarrow 0} \frac{\tan x}{x} =$

- A)  $-1$
  - B)  $-\frac{1}{2}$
  - C)  $0$
  - D)  $\frac{1}{2}$
  - E)  $1$
- 

9. Suppose  $f$  is defined as

$$f(x) = \begin{cases} \frac{|x| - 2}{x - 2} & x \neq 2 \\ k & x = 2. \end{cases}$$

Then the value of  $k$  for which  $f(x)$  is continuous for all real values of  $x$  is  $k =$

- A)  $-2$
- B)  $-1$
- C)  $0$
- D)  $1$
- E)  $2$

10. The average rate of change of  $f(x) = x^3$  over the interval  $[a, b]$  is

- A)  $3b + 3a$
  - B)  $b^2 + ab + a^2$
  - C)  $\frac{b^2 + a^2}{2}$
  - D)  $\frac{b^3 - a^3}{2}$
  - E)  $\frac{b^4 - a^4}{4(b - a)}$
- 

11. The function

$$G(x) = \begin{cases} x - 5 & x > 2 \\ -5 & x = 2 \\ 5x - 13 & x < 2 \end{cases}$$

is not continuous at  $x = 2$  because

- A)  $G(2)$  is not defined.
  - B)  $\lim_{x \rightarrow 2} G(x)$  does not exist.
  - C)  $\lim_{x \rightarrow 2} G(x) \neq G(2)$ .
  - D)  $G(2) \neq -5$ .
  - E) None of the above
- 

12.  $\lim_{x \rightarrow -2} \frac{\sqrt{2x+5} - 1}{x+2} =$

- A) 1
- B) 0
- C)  $\infty$
- D)  $-\infty$
- E) does not exist

**13.** The Intermediate Value Theorem states that given a continuous function  $f$  defined on the closed interval  $[a, b]$  for which 0 is between  $f(a)$  and  $f(b)$ , there exists a point  $c$  between  $a$  and  $b$  such that

- A)  $c = a - b$
  - B)  $f(a) = f(b)$
  - C)  $f(c) = 0$
  - D)  $f(0) = c$
  - E)  $c = 0$
- 

**14.** The function  $t(x) = 2^x - \frac{|x-3|}{x-3}$  has

- A) a removable discontinuity at  $x = 3$ .
  - B) an infinite discontinuity at  $x = 3$ .
  - C) a jump discontinuity at  $x = 3$ .
  - D) no discontinuities.
  - E) a removable discontinuity at  $x = 0$  and an infinite discontinuity at  $x = 3$ .
- 

**15.** Find the values of  $c$  so that the function

$$h(x) = \begin{cases} c^2 - x^2 & x < 2 \\ x + c & x \geq 2 \end{cases}$$

is continuous everywhere.

- A)  $-3, -2$
- B)  $2, 3$
- C)  $-2, 3$
- D)  $-3, 2$
- E) There are no such values.

**A.P. Calculus Test One**  
**Section Two**  
**Free-Response**  
**Calculators Allowed**  
Time—45 minutes  
Number of Questions—3

Each of the three questions is worth 9 points. The maximum possible points earned on this section is 27, which represents 50% of the total test score. There is no penalty for guessing.

- **SHOW ALL YOUR WORK.** You will be graded on the methods you use as well as the accuracy of your answers. Correct answers without supporting work may not receive full credit.
- Write all work for each problem in the space provided. Be sure to write clearly and legibly. Erased or crossed out work will not be graded.
- Justifications require that you give mathematical (non-calculator) reasons and that you clearly identify functions, graphs, tables, or other objects that you use.
- You are permitted to use your calculator to solve an equation or graph a function without showing work. However, you must clearly indicate the setup of your problem.
- Your work must be expressed in mathematical notation rather than calculator syntax. For example,  $y = x^2$  may not be written as  $Y1=X^2$ .
- Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Good Luck!

NAME:

1. Consider the function  $f(x) = \frac{|x|(x-3)}{9-x^2}$ .

- What is the domain of  $f$ ? What are the zeros of  $f$ ?
  - Evaluate  $\lim_{x \rightarrow 3} f(x)$ .
  - Determine all vertical and horizontal asymptotes of  $f$ .
  - Find all the nonremovable discontinuities of  $f$ .
- 

2. Consider the function  $g(x) = x^x$  with domain  $(0, \infty)$ .

- a) Fill in the following table.

$x$	0.01	0.1	0.2	0.3	0.4	0.5	1
$x^x$							

- What is  $\lim_{x \rightarrow 1^-} g(x)$ ? What is  $\lim_{x \rightarrow 0^+} g(x)$ ?
  - What do you think the smallest value of  $g(x)$  is for values in the interval  $(0, 1)$ ? Justify your answer.
  - Find the average rate of change of  $g(x)$  from  $x = 0.1$  to  $x = 0.4$ .
- 

3. Consider the function  $F(x) = (a^{-1} - x^{-1})^{-1}$  where  $a$  is a positive real number.

- What is the domain of  $F$ ? What are the zeros of  $F$ ?
- Find all asymptotes of  $F$  and discuss any discontinuities of  $F$ .
- Evaluate  $\lim_{x \rightarrow 0} F(x)$ ,  $\lim_{x \rightarrow \infty} F(x)$ , and  $\lim_{x \rightarrow a} F(x)$ .
- For what value of  $a$  will  $F(6) = 12$ ?

## CHAPTER 2

---

# DERIVATIVES

---

## 2.1 Negative and Fractional Exponents

REWRITE EACH EXPRESSION WITH FRACTIONAL EXPONENTS AND SIMPLIFY.

332.  $\sqrt[3]{x} \sqrt[5]{y^2}$

334.  $x^3 \sqrt[5]{x^3}$

333.  $\sqrt{x+2} \sqrt[4]{(x+2)^9}$

335.  $(x+6)^4 \sqrt[3]{x+6}$

REWRITE EACH EXPRESSION WITH RADICALS AND SIMPLIFY.

336.  $x^{5/3}$

339.  $16^{7/4}$

337.  $8(x+2)^{5/2}$

340.  $(64x)^{3/2}$

338.  $y^{10/3}$

REWRITE AND SIMPLIFY EACH OF THE FOLLOWING IN TWO WAYS: A) WITH POSITIVE EXPONENTS ONLY; AND B) WITH NO DENOMINATORS.

341.  $\frac{x^2 y^{-3}}{x^{-4} y^2}$

343.  $\frac{(x+5)^{-2}(x+7)^3}{(x+7)^4(x+5)^3}$

342.  $\frac{x^{-2/5} y^{-3/4}}{x^{-3/5} y^{1/4}}$

344.  $x^2(x^{-2/3} + x^{-7/3})$

COMPLETELY FACTOR EACH OF THE FOLLOWING EXPRESSIONS.

345.  $2x^{3/5} - 4x^{1/5}$

346.  $8x^{10/3} + 16x^{5/3} + 8$

347.  $25x^{6/5} - 49x^{8/3}$

348.  $4x^{-7/3} - 6x^{-5/3} + 12x^{-1}$

349.  $x^3 + x^2 - x^{-2} - x^{-3}$

350.  $(\frac{4}{3}x^{4/3} + 2x)(x^{2/3} + 4x^{1/3})$

351.  $\frac{1}{2}(x^3 + 3x^2)^{-1/2}(2x + 4)$

352.  $(x^2 + 6x + 9)^{-1/2}(x + 3)^{3/2}$

353.  $(x^{-1/3} + x^{-2/3})(x^{1/3} + 1) + (x^{2/3} + 3x^{1/3} + 2)$

354.  $\frac{\frac{2}{3}(x-2)^{-1/3}x^{4/3} - \frac{4}{3}(x-2)^{2/3}x^{1/3}}{x^{8/3}}$

355.  $\frac{\frac{1}{2}(x^2 + 7)^{-1/2}2x\sqrt{x} - \frac{1}{2}x^{-1/2}\sqrt{x^2 + 7}}{x}$

356.  $\frac{\frac{1}{2}(x-7)^{-1/2}(x-3) - \sqrt{x-7}}{(x-3)^2}$



## 2.2 Logically Thinking About Logic

IN EACH OF THE FOLLOWING PROBLEMS, YOU ARE GIVEN A TRUE STATEMENT. FROM THE STATEMENT, DETERMINE WHICH ONE OF THE THREE CHOICES IS LOGICALLY EQUIVALENT. (YOU DO NOT NEED TO KNOW WHAT THE WORDS MEAN IN ORDER TO DETERMINE THE CORRECT ANSWER.)

**357.** If it is raining, then I will go to the mall.

- A) If I go to the mall, then it is raining.
- B) If it is not raining, then I will not go to the mall.
- C) If I do not go to the mall, then it is not raining.

**358.** If a snark is a grunk, then a quango is a trone.

- A) If a quango is a trone, then a snark is a grunk.
- B) If a quango is not a trone, then a snark is not a grunk.
- C) If a snark is not a grunk, then a quango is not a trone.

**359.** If a function is linear, then the graph is not a parabola.

- A) If the graph is a parabola, then the function is not linear.
- B) If the graph is a parabola, then the function is linear.
- C) If the function is not linear, then the graph is a parabola.

**360.** If a function has a vertical asymptote, then it is either rational, logarithmic, or trigonometric.

- A) If a function is rational, logarithmic, or trigonometric, then the function has a vertical asymptote.
- B) If a function is not rational, logarithmic, and trigonometric, then the function has no vertical asymptote.
- C) If a function is neither rational, logarithmic, and trigonometric, then the function has no vertical asymptote.

**361.** If  $f(x)$  is continuous and  $f(a) = f(b)$ , then there is a number  $c$  between  $a$  and  $b$  so that  $f(c)$  is the maximum of  $f(x)$ .

- A) If  $f(x)$  is not continuous and  $f(a) = f(b)$ , then there is not a number  $c$  between  $a$  and  $b$  so that  $f(c)$  is the maximum of  $f(x)$ .
- B) If there is a number  $c$  between  $a$  and  $b$  so that  $f(c)$  is not the maximum of  $f(x)$ , then either  $f(x)$  is not continuous or  $f(a) \neq f(b)$ .
- C) If there is not a number  $c$  between  $a$  and  $b$  so that  $f(c)$  is the maximum of  $f(x)$ , then  $f(x)$  is not continuous or  $f(a) \neq f(b)$ .

### 2.3 The Derivative By Definition

FOR EACH OF THE FOLLOWING, USE THE DEFINITION OF THE DERIVATIVE TO A) FIND AN EXPRESSION FOR  $f'(x)$  AND B) FIND THE VALUE OF  $f'(a)$  FOR THE GIVEN VALUE OF  $a$ .

**362.**  $f(x) = 2x - 3$ ;  $a = 0$

**364.**  $f(x) = \sqrt{1 + 2x}$ ;  $a = 4$

**363.**  $f(x) = x^2 - x$ ;  $a = 1$

**365.**  $f(x) = \frac{1}{x}$ ;  $a = 2$

DIFFERENTIATE EACH FUNCTION. YOU DO NOT NEED TO USE THE DEFINITION.

**366.**  $g(x) = 3x^2 - 2x + 1$

**371.**  $k(x) = (x^{1/3} - 2)(x^{2/3} + 2x^{1/3} + 4)$

**367.**  $p(x) = (x - 1)^3$

**372.**  $y(x) = x^2 - 3x - 5x^{-1} + 7x^{-2}$

**368.**  $w(x) = (3x^2 + 4)^2$

**373.**  $G(x) = (3x - 1)(2x + 5)$

**369.**  $J(x) = \frac{3x^4 - 2x^3 + 6x}{12x}$

**374.**  $S(x) = \sqrt{x} + 17\sqrt[3]{x^2}$

**370.**  $t(x) = \frac{5}{2x^3} - \frac{3}{5x^4}$

**375.**  $V(x) = \frac{2}{3}\pi x^3 + 10\pi x^2$

ANSWER EACH OF THE FOLLOWING.

**376.** What is the derivative of any function of the form  $y = a$ , where  $a$  is any constant?

**377.** What is the derivative of any function of the form  $y = mx + b$ , where  $m$  and  $b$  are any constants?

**378.** What is the derivative of any function of the form  $y = x^n$ , where  $n$  is any constant?

**379.** If  $3x^2 + 6x - 1$  is the derivative of a function, then what could be the original function?

**380.** Let  $y = 7x^2 - 3$ . Find  $y'$  and  $y'(1)$ . Find  $\frac{dy}{dx}$  and  $\frac{dy}{dx}\Big|_{x=2}$ .

DETERMINE IF EACH OF THE FOLLOWING FUNCTIONS IS DIFFERENTIABLE AT  $x = 1$ ; THAT IS, DOES THE DERIVATIVE EXIST AT  $x = 1$ ?

**381.**  $f(x) = |x - 1|$

**384.**  $f(x) = \begin{cases} x & x \leq 1 \\ x^2 & x > 1 \end{cases}$

**382.**  $f(x) = \sqrt{1 - x^2}$

**385.**  $f(x) = \begin{cases} x^2 & x \leq 1 \\ 4x - 2 & x > 1 \end{cases}$

**383.**  $f(x) = \begin{cases} (x - 1)^3 & x \leq 1 \\ (x - 1)^2 & x > 1 \end{cases}$

**386.**  $f(x) = \begin{cases} \frac{1}{2}x & x < 1 \\ \sqrt{x} - 1 & x \geq 1 \end{cases}$

---

A habit of basing convictions upon evidence, and of giving to them only that degree of certainty which the evidence warrants, would, if it became general, cure most of the ills from which the world suffers. —*Bertrand Russell*

## 2.4 Going Off on a Tangent

FOR THE FOLLOWING FIVE PROBLEMS, FIND AN EQUATION FOR THE TANGENT LINE TO THE CURVE AT THE GIVEN  $x$ -COORDINATE.

**387.**  $y = 4 - x^2$ ;  $x = -1$

**388.**  $y = 2\sqrt{x}$ ;  $x = 1$

**389.**  $y = x - 2x^2$ ;  $x = 1$

**390.**  $y = x^{-3}$ ;  $x = -2$

**391.**  $y = x^3 + 3x$ ;  $x = 1$

**392.** At what points does the graph of  $y = x^2 + 4x - 1$  have a horizontal tangent?

**393.** Find an equation for the tangent to the curve  $y = \sqrt{x}$  that has slope  $\frac{1}{4}$ .

**394.** What is the instantaneous rate of change of the area of a circle when the radius is 3 cm?

**395.** What is the instantaneous rate of change of the volume of a ball when the radius is 2 cm?

**396.** Does the graph of  $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$  have a tangent at the origin? Justify your answer.

**397.** Consider the curve  $y = x^3 - 4x + 1$ .

- Find an equation for the tangent to the curve at the point  $(2, 1)$ .
- What is the range of values of the curve's slope?
- Find equations for the tangents to the curve at the points where the slope of the curve is 8.

DETERMINE WHICH OF THE FOLLOWING FUNCTIONS ARE DIFFERENTIABLE AT  $x = 0$ .

**398.**  $y = x^{1/3}$

**402.**  $y = x^{1/4}$

**399.**  $y = x^{2/3}$

**403.**  $y = x^{5/4}$

**400.**  $y = x^{4/3}$

**404.**  $y = x^{1/5}$

**401.**  $y = x^{5/3}$

**405.**  $y = x^{2/5}$

**406.** Based on the answers from the problems above, find a pattern for the differentiability of functions with exponents of the following forms:  $x^{\text{even/odd}}$ ,  $x^{\text{odd/odd}}$ ,  $x^{\text{odd/even}}$ .

## 2.5 Six Derivative Problems

**407.** Water is flowing into a large spherical tank at a constant rate. Let  $V(t)$  be the volume of water in the tank at time  $t$ , and  $h(t)$  be the height of the water level at time  $t$ .

- Give a physical interpretation of  $\frac{dV}{dt}$  and  $\frac{dh}{dt}$ .
- Which of  $\frac{dV}{dt}$  and  $\frac{dh}{dt}$  is constant? Explain your answer.
- Is  $\frac{dV}{dt}$  positive, negative, or zero when the tank is one quarter full?
- Is  $\frac{dh}{dt}$  positive, negative, or zero when the tank is one quarter full?

**408.** Let  $f(x) = 2^x$ .

- Find the average rate of change of  $f$  from  $x = -1$  to  $x = 1$ .
- Find the average rate of change of  $f$  from  $x = -\frac{1}{2}$  to  $x = \frac{1}{2}$ .
- Use your calculator to estimate  $f'(0)$ , the instantaneous rate of change of  $f$  at 0.
- Sketch the graph of  $f$  and use it to explain why the answer to part (b) is a better estimate of  $f'(0)$  than the answer to part (a). Can you suggest a generalization of your ideas?

**409.** The position  $p(t)$  of an object at time  $t$  is given by  $p(t) = 3t^2 + 1$ .

- Find the instantaneous velocity of the object at an arbitrary time  $t$ .
- Find the instantaneous velocity of the object at time  $t = -1$ .

**410.** Let  $f(x) = x^2 + x - 2$ .

- Use the definition of the derivative to find  $f'(x)$ .
- Find an equation of the tangent line to the graph of  $f$  at the point  $(-1, -2)$ .
- Sketch the graph of  $f$  together with the tangent line found in part (b) on the same axes.

**411.** Find a function  $f(x)$  and a point  $a$  such that  $f'(a)$  does not exist even though  $f(a)$  does.

**412.** There's dust on my guitar! The total amount of dust after  $t$  days is given by  $g(t)$ . I know that  $g(30) = 270$  milligrams and  $g'(30) = 5$ .

- Estimate  $g(32)$ .
- What are the units of  $g'(t)$ ?

---

Many very learned men have no intelligence. —*Democritus*

Nothing is more terrible than to see ignorance in action. —*Johann Wolfgang von Goethe*

## 2.6 Trigonometry: a Refresher

EVALUATE EACH OF THE FOLLOWING EXPRESSIONS. DO NOT USE A CALCULATOR.

- |  |  |   |
|--|--|---|
| <b>413.</b> $\tan \frac{\pi}{4}$                         | <b>418.</b> $\sin^2 \frac{5\pi}{6} + \tan^2 \frac{\pi}{6}$ | <b>423.</b> $\sin(\arctan 1)$               |
| <b>414.</b> $(\sin \frac{3\pi}{4})(\cos \frac{5\pi}{4})$ | <b>419.</b> $\arcsin \frac{1}{2}$                          | <b>424.</b> $\tan(\sec^{-1} 2)$             |
| <b>415.</b> $\sec \frac{4\pi}{3}$                        | <b>420.</b> $\arctan \frac{1}{\sqrt{3}}$                   | <b>425.</b> $\sin(\arcsin 0.3)$             |
| <b>416.</b> $\cos(-\frac{\pi}{4})$                       | <b>421.</b> $\sin^{-1}(-\frac{\sqrt{3}}{2})$               | <b>426.</b> $\arcsin(\sin \pi)$             |
| <b>417.</b> $\sin(\frac{\pi}{2} - \frac{\pi}{6})$        | <b>422.</b> $\tan^{-1}(-\sqrt{3})$                         | <b>427.</b> $\arccos(\cos(-\frac{\pi}{4}))$ |

**428.** Which of the following are undefined?

- a)  $\arccos 1.5$       b)  $\operatorname{arcsec} 1.5$       c)  $\arctan 1.5$       d)  $\operatorname{arcsec} 0.3$       e)  $\arcsin 2.4$

EVALUATE THE FOLLOWING LIMITS. GRAPH THE FUNCTIONS ON YOUR CALCULATOR IF NECESSARY.

- |  |   |   |
|--|---|---|
| <b>429.</b> $\lim_{x \rightarrow 1^+} \sin^{-1} x$ | <b>431.</b> $\lim_{x \rightarrow 1} \csc^{-1} x$    | <b>433.</b> $\lim_{x \rightarrow -\infty} \arctan x$              |
| <b>430.</b> $\lim_{x \rightarrow 1} \sec^{-1} x$   | <b>432.</b> $\lim_{x \rightarrow \infty} \arctan x$ | <b>434.</b> $\lim_{x \rightarrow \infty} \operatorname{arcsec} x$ |

**435.** We know  $\sin x$  is an odd function and  $\cos x$  is an even function, but what about these?

- a)  $\arccos x$       b)  $\arcsin x$       c)  $\arctan x$       d)  $\sec x$       e)  $\csc x$

FIND EXACT SOLUTIONS TO EACH OF THE FOLLOWING EQUATIONS OVER THE INTERVAL  $[0, 2\pi)$ .

- |   |   |
|---|---|
| <b>436.</b> $\cos 3\theta - 1 = 0$                  | <b>439.</b> $2\sin^2 \theta - 3\sin \theta + 1 = 0$ |
| <b>437.</b> $\tan 2x + 1 = 0$                       | <b>440.</b> $2\cos^2 \theta + \cos \theta = 0$      |
| <b>438.</b> $\sin 3\theta + \frac{\sqrt{2}}{2} = 0$ | <b>441.</b> $\cos x + 2\sec x = -3$                 |

**442.** Water is draining from a tank. The volume of water in the tank is given by  $V(t) = 1000 + (20 - t)^3$ , where  $V$  is in gallons and  $t$  is the number of hours since the water began draining. Answer the following questions using correct units.

- a) How much water is in the tank initially?
- b) How fast is it draining after 10 hours?
- c) Will the tank have been completely drained after two days? Why?

## 2.7 Continuity and Differentiability

**443 (AP).** Suppose  $f$  is a function for which  $\lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} = 0$ . Which of the following must be true, might be true, or can never be true?

- a)  $f'(2) = 2$
- b)  $f(2) = 0$
- c)  $\lim_{x \rightarrow 2} f(x) = f(2)$
- d)  $f(x)$  is continuous at  $x = 0$ .
- e)  $f(x)$  is continuous at  $x = 2$ .

**444 (AP).** For some nonzero real number  $a$ , define the function  $f$  as  $f(x) = \begin{cases} \frac{x^2 - a^2}{x - a} & x \neq a \\ 0 & x = a. \end{cases}$

- a) Is  $f$  defined at  $a$ ?
- b) Does  $\lim_{x \rightarrow a} f(x)$  exist? Justify your answer.
- c) Is  $f$  continuous at  $a$ ? Justify your answer.
- d) Is  $f$  differentiable at  $a$ ? Justify your answer.

**445.** If  $\lim_{x \rightarrow a} f(x) = L$ , which of the following statements, if any, *must* be true? Justify your answers.

- a)  $f$  is defined at  $a$ .
- b)  $f(a) = L$ .
- c)  $f$  is continuous at  $a$ .
- d)  $f$  is differentiable at  $a$ .

**446.** Let  $f(x) = \begin{cases} ax & x \leq 1 \\ bx^2 + x + 1 & x > 1. \end{cases}$

- a) Find all choices of  $a$  and  $b$  such that  $f$  is continuous at  $x = 1$ .
- b) Draw the graph of  $f$  when  $a = 1$  and  $b = -1$ .
- c) Find the values of  $a$  and  $b$  such that  $f$  is differentiable at  $x = 1$ .
- d) Draw the graph of  $f$  for the values of  $a$  and  $b$  found in part (c).

---

Don't just read it; fight it! Ask your own questions, look for your own examples, discover your own proofs. Is the hypothesis necessary? Is the converse true? What happens in the classical special case? What about the degenerate cases? Where does the proof use the hypothesis? —*Jacques Hadamard*

## 2.8 The RULES: Power Product Quotient Chain

**447.** Let  $f(x) = \begin{cases} 3 - x & x < 1 \\ ax^2 + bx & x \geq 1 \end{cases}$  where  $a$  and  $b$  are constants.

- a) If the function is continuous for all  $x$ , what is the relationship between  $a$  and  $b$ ?  
 b) Find the unique values for  $a$  and  $b$  that will make  $f$  both continuous and differentiable.

**448.** Suppose that  $u(x)$  and  $v(x)$  are differentiable functions of  $x$  and that

$$u(1) = 2, \quad u'(1) = 0, \quad v(1) = 5, \quad \text{and} \quad v'(1) = -1.$$

Find the values of the following derivatives at  $x = 1$ .

a)  $\frac{d}{dx}(uv)$                       b)  $\frac{d}{dx}\left(\frac{u}{v}\right)$                       c)  $\frac{d}{dx}\left(\frac{v}{u}\right)$                       d)  $\frac{d}{dx}(7v - 2u)$

**449.** Graph the function  $y = \frac{4x}{x^2 + 1}$  on your calculator in the window  $-5 \leq x \leq 5$ ,  $-3 \leq y \leq 3$ . (This graph is called *Newton's serpentine*.) Find the tangent lines at the origin and at the point  $(1, 2)$ .

**450.** Graph the function  $y = \frac{8}{x^2 + 4}$  on your calculator in the window  $-5 \leq x \leq 5$ ,  $-3 \leq y \leq 3$ . (This graph is called the *witch of Agnesi*.) Find the tangent line at the point  $(2, 1)$ .

FIND THE DERIVATIVE OF THE GIVEN FUNCTION. EXPRESS YOUR ANSWER IN SIMPLEST FACTORED FORM.

**451.**  $A(z) = (3z - 5)^4$

**460.**  $h(u) = \sqrt{u - 1} \sqrt[3]{2u + 3}$

**452.**  $q(u) = (3u^5 - 2u^3 - 3u - \frac{1}{3})^3$

**461.**  $f(x) = \frac{3x}{x + 5}$

**453.**  $b(y) = (y^3 - 5)^{-4}$

**462.**  $g(y) = \frac{4y - 3}{3 - 2y}$

**454.**  $c(d) = \sqrt[3]{(5d^2 - 1)^5}$

**463.**  $p(x) = \frac{x^2 + 10x + 25}{x^2 - 10x + 25}$

**455.**  $u(p) = \frac{3p^2 - 5}{p^3 + 2p - 6}$

**464.**  $m(x) = \frac{7x}{1 - 3x}$

**456.**  $V(x) = \frac{\sqrt{5x^3}}{5x^3}$

**465.**  $f(x) = \frac{3}{x^2} - \frac{x^2}{3}$

**457.**  $f(x) = 3x^{1/3} - 5x^{-1/3}$

**466.**  $g(x) = \left(\frac{4x - 3}{5 - 3x}\right)(2x + 7)$

**458.**  $g(z) = \frac{1}{\sqrt{36 - z^2}}$

**467.**  $F(x) = 10x^{27} - 25x^{1/5} + 12x^{-12} + 350$

**459.**  $p(t) = (3 - 2t)^{-1/2}$

---

A man is like a fraction whose numerator is what he is and whose denominator is what he thinks of himself. The larger the denominator, the smaller the fraction. —*Leo Tolstoy*

## 2.9 Trigonometric Derivatives

FIND  $\frac{dy}{dx}$  FOR EACH OF THE FOLLOWING.

468.  $y = 3 \cos x$

475.  $y = \sin \sqrt{x}$

469.  $y = \cot x$

476.  $y = \cos(3x + 1)$

470.  $y = \tan x - x$

477.  $y = \sin^2(4x)$

471.  $y = x \sin x + \cos x$

478.  $y = 2 \sin x \cos x$

472.  $y = \sin\left(\frac{3\pi x}{2}\right)$

479.  $y = \pi \cot(\pi x)$

473.  $y = \cos^2 x$

480.  $y = x^2 \tan x$

474.  $y = \tan^3 x$

481.  $y = 8 \csc 8x$

482. Find all points on the curve  $y = \tan x$  over the interval  $\frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$  where the tangent line is parallel to the line  $y = 2x$ .

483. Graph  $y = 1 + \sqrt{2} \csc x + \cot x$  on your calculator in the window  $0 \leq x \leq \pi$ ,  $-1 \leq y \leq 9$ . Find the equation of the tangent line at the point  $(\frac{\pi}{4}, 4)$ ; then find the point on the graph where the graph has a horizontal tangent.

484. Is there a value of  $b$  that will make  $g(x) = \begin{cases} x + b & x < 0 \\ \cos x & x \geq 0 \end{cases}$  continuous at  $x = 0$ ? Differentiable at  $x = 0$ ? Justify your answers.

485. Find the 1000th derivative of  $\cos x$ .

486. Find the tangent to the curve  $y = 2 \tan\left(\frac{\pi x}{4}\right)$  at  $x = 1$ .

FIND  $y''$  FOR EACH OF THE FOLLOWING.

487.  $y = \csc \theta$

488.  $y = \sec \theta$

489.  $y = 2 - 2 \sin \theta$

490.  $y = \sin \theta + \cos \theta$

---

Neither in the subjective nor in the objective world can we find a criterion for the reality of the number concept, because the first contains no such concept, and the second contains nothing that is free from the concept. How then can we arrive at a criterion? Not by evidence, for the dice of evidence are loaded. Not by logic, for logic has no existence independent of mathematics: it is only one phase of this multiplied necessity that we call mathematics. How then shall mathematical concepts be judged? They shall not be judged. Mathematics is the supreme arbiter. From its decisions there is no appeal. We cannot change the rules of the game, we cannot ascertain whether the game is fair. We can only study the player at his game; not, however, with the detached attitude of a bystander, for we are watching our own minds at play. —*Dantzig*



## 2.10 Tangents, Normals, and Continuity (Revisited)

- 491.** Find the equation of the tangent line to the curve  $y = \sqrt{x^2 - 3}$  at the point  $(2, 1)$ .
- 492.** Find the equation of the normal line to the curve  $y = (3x - 1)^2(x - 1)^3$  at  $x = 0$ .
- 493.** Find the equation of the tangent line to the curve  $y = \sqrt{3x - 1}$  that is perpendicular to the line  $3y + 2x = 3$ .
- 494.** Find the equation of the normal line to the curve  $y = x\sqrt{25 + x^2}$  at  $x = 0$ .
- 495.** Find the equation of the tangent line to the curve  $y = \frac{2 - x}{5 + x}$  at  $x = 1$ .
- 496.** Find the equation of the normal line to the curve  $y = \frac{5}{(5 - 2x)^2}$  at  $x = 0$ .
- 497.** Find the equation of the tangent line to the curve  $y = 3x^4 - 2x + 1$  that is parallel to the line  $y - 10x - 3 = 0$ .
- 498.** The point  $P(3, -2)$  is not on the graph of  $y = x^2 - 7$ . Find the equation of each line tangent to  $y = x^2 - 7$  that passes through  $P$ .

FOR THE FOLLOWING SIX PROBLEMS, DETERMINE IF  $f$  IS DIFFERENTIABLE AT  $x = a$ .

**499.**  $f(x) = |x + 5|$ ;  $a = -5$

**502.**  $f(x) = \begin{cases} -2x^2 & x < 0 \\ 2x^2 & x \geq 0 \end{cases}$   $a = 0$

**500.**  $f(x) = \begin{cases} x + 3 & x \leq -2 \\ -x - 1 & x > -2 \end{cases}$   $a = -2$

**503.**  $f(x) = \begin{cases} x^2 - 5 & x < 3 \\ 3x - 5 & x \geq 3 \end{cases}$   $a = 3$

**501.**  $f(x) = \begin{cases} 2 & x < 0 \\ x - 4 & x \geq 0 \end{cases}$   $a = 0$

**504.**  $f(x) = \begin{cases} \sqrt{2 - x} & x < 2 \\ (2 - x)^2 & x \geq 2 \end{cases}$   $a = 2$

- 505.** Suppose that functions  $f$  and  $g$  and their first derivatives have the following values at  $x = -1$  and at  $x = 0$ .

$x$	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
-1	0	-1	2	1
0	-1	-3	-2	4

Evaluate the first derivatives of the following combinations of  $f$  and  $g$  at the given value of  $x$ .

a)  $3f(x) - g(x)$ ,  $x = -1$

d)  $f(g(x))$ ,  $x = -1$

b)  $[f(x)]^3[g(x)]^3$ ,  $x = 0$

e)  $\frac{f(x)}{g(x) + 2}$ ,  $x = 0$

c)  $g(f(x))$ ,  $x = -1$

f)  $g(x + f(x))$ ,  $x = 0$

## 2.11 Implicit Differentiation

FIND  $\frac{dy}{dx}$  FOR EACH OF THE FOLLOWING.

506.  $x^2 - y^2 = 5$

509.  $x = \tan y$

507.  $1 - xy = x - y$

510.  $x^3 - xy + y^3 = 1$

508.  $y^2 = x^3$

511.  $9x^2 + 25y^2 = 225$

512. Find the equation of both the tangent and normal lines to the curve  $x^2 \cos^2 y - \sin y = 0$  at the point  $(0, \pi)$ .

513. Find the equation of both the tangent and normal lines to the curve  $y^2(2 - x) = x^3$  at the point  $(1, 1)$ .

FIND  $\frac{d^2y}{dx^2}$  IN TERMS OF  $x$  AND  $y$  FOR THE FOLLOWING THREE PROBLEMS.

514.  $xy + y^2 = 1$

515.  $y^2 = x^2 + 2x$

516.  $x^2 + xy = 5$

517. Find the equation of the tangent line to the curve  $(x^2 + y^2)^2 = 4x^2y$  at the point  $(1, 1)$ .

518. Consider the curve defined by  $x^3 + y^3 - 9xy = 0$ .

- Find the equation of the tangent lines at the points  $(4, 2)$  and  $(2, 4)$ .
- At what points does the curve have a horizontal tangent?
- Find the coordinates of the point where the curve has a vertical tangent.

519. Find the two points where the curve  $x^2 + xy + y^2 = 7$  crosses the  $x$ -axis and show that the tangents to the curve at these points are parallel. What is the common slope of these tangents?

520. The line that is normal to the curve  $x^2 + 2xy - 3y^2 = 0$  at the point  $(1, 1)$  intersects the curve at what other point?

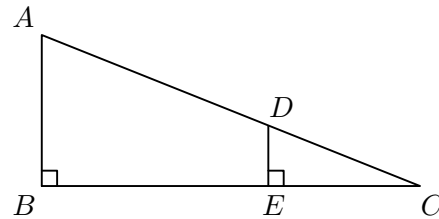
521 (AP, 2000AB). Consider the curve given by  $xy^2 - x^3y = 6$ .

- Find  $\frac{dy}{dx}$ .
- Find all points on the curve whose  $x$ -coordinate is 1, and write an equation for the tangent line at each of these points.
- Find the  $x$ -coordinate of each point on the curve where the tangent is vertical.

## 2.12 The Return of Geometry

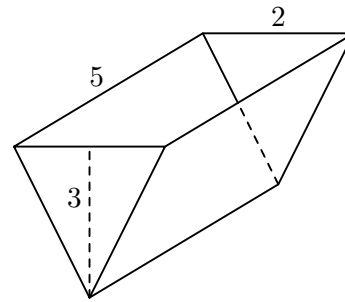
- 522.** Find the area and circumference of a circle of radius 7.
- 523.** Find the volume of a cylinder with radius 8 and height 10.
- 524.** Find the volume and surface area of a sphere of radius 9.
- 525.** Find the volume and surface area of a cube of side length 6.
- 526.** Find the volume and surface area of a box with dimensions 3, 4, and 5.
- 527.** What is the hypotenuse of a right triangle with legs 5 and 12?
- 528.** The area of an isosceles right triangle is 8. What is the length of its hypotenuse?
- 529.** A cylinder is constructed so that its height is exactly 4 times its radius. If the volume of the cylinder is  $500\pi$ , then what is its radius?

- 530.** In the figure to the right,  $DE = 2$ ,  $EC = 5$ , and  $AB = 5$ . Find the lengths of  $\overline{AC}$  and  $\overline{BC}$ .



- 531.** What is the area of an equilateral triangle if its side lengths are 8?
- 532.** What is the area of a semicircle of radius 10?

- 533.** The trough shown in the figure at the right is 5 feet long and its vertical cross sections are inverted isosceles triangles with base 2 feet and height 3 feet. Find the volume of water in the trough when the trough is full.



- 534.** A cone is constructed so that its height is exactly 4 times its radius. If the volume of the cone is  $324\pi$ , then what is its radius?
- 535.** A 12-foot ladder is leaning against a wall so that it makes a  $60^\circ$  angle with the ground. How high up the wall does the ladder touch the wall?
- 536.** An equilateral triangle has an area of  $4\sqrt{3}$ . What is the height of this equilateral triangle?

## 2.13 Meet the Rates (They're Related)

SOLVE THE FOLLOWING PROBLEMS, ASSUMING THAT ALL VARIABLES ARE FUNCTIONS OF THE VARIABLE  $t$ .

**537.** If  $xy = -3$  and  $\frac{dx}{dt} = 1$ , then find  $\frac{dy}{dt}$  when  $x = 6$ .

**538.** If  $x^2 - y^2 = 39$  and  $\frac{dx}{dt} = 2$ , then find  $\frac{dy}{dt}$  when  $y = 5$ .

**539.** If  $\frac{y}{z} = 13$  and  $\frac{dz}{dt} = -2$ , then find  $\frac{dy}{dt}$  when  $y = 26$ .

SOLVE EACH OF THE FOLLOWING PROBLEMS.

**540.** The volume of a cube is decreasing at the rate of  $10 \text{ m}^3/\text{hr}$ . How fast is the total surface area decreasing when the surface area is  $54 \text{ m}^2$ ?

**541.** The length  $l$  of a rectangle is decreasing at the rate of  $2 \text{ cm/sec}$  while the width  $w$  is increasing at the rate of  $2 \text{ cm/sec}$ . When  $l = 12 \text{ cm}$  and  $w = 5 \text{ cm}$ , find the rates of change of a) the area; b) the perimeter; and c) the length of the diagonal of the rectangle. Which of these quantities are decreasing and which are increasing?

**542.** Rachael is blowing up a balloon so that the diameter increases at the rate of  $10 \text{ cm/sec}$ . At what rate must she blow air into the balloon when the diameter measures  $4 \text{ cm}$ ?

**543.** Assume Clark and Lana leave Smallville Stadium from the same point at the same time. If Clark runs south at  $4 \text{ mph}$  and Lana runs west at  $3 \text{ mph}$ , how fast will the distance between Clark and Lana be changing at  $10 \text{ hours}$ ?

**544.** Suppose Aaron is pumping water into a tank (in the shape of an inverted right circular cone) at a rate of  $1600 \text{ ft}^3/\text{min}$ . If the altitude is  $10 \text{ ft}$  and the radius of the base is  $5 \text{ ft}$ , find the rate at which the radius is changing when the height of the water is  $7 \text{ ft}$ .

**545.** LuthorCorp Industries hires Professor Patel to calculate the revenue and cost of their best-selling pesticide. Professor Patel finds that the revenue is  $R(x) = 700x - \frac{x^2}{5000}$  and the cost is  $C(x) = 300 + 4x$ , where  $x$  is the number of gallons of pesticide produced each week. If the rate of production is increasing by  $50$  gallons per week, and the current production is  $300$  gallons per week, find the rate of change in a) the revenue  $R$ ; b) the cost  $C$ ; and c) the profit  $P = R - C$ .

**546.** The area of an equilateral triangle is increasing at the rate of  $5 \text{ m}^2/\text{hr}$ . Find the rate at which the height is changing when the area is  $\frac{64}{\sqrt{3}} \text{ m}^2$ .

**547.** The talented Ed Wynwyte is flying a kite at a constant height of  $400 \text{ meters}$ . The kite is moving horizontally at a rate of  $30 \text{ m/sec}$ . How fast must he unwind the string when the kite is  $500 \text{ m}$  away from him?

## 2.14 Rates Related to the Previous Page

**548.** A ladder 15 feet tall leans against a vertical wall of a home. If the bottom of the ladder is pulled away horizontally from the house at 4 ft/sec, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 9 feet from the wall?

**549.** A cone (vertex down) with height 10 inches and radius 2 inches is being filled with water at a constant rate of 2 in<sup>3</sup>/sec. How fast is the surface of the water rising when the depth of the water is 6 inches?

**550.** A particle is moving along the graph of  $y = \sqrt{x}$ . At what point on the curve are the  $x$ -coordinate and  $y$ -coordinate of the particle changing at the same rate?

**551.** A streetlight is 15 feet above the sidewalk. Jonathan, who is 7 feet tall, walks away from the light at the rate of 5 feet per second.

- Determine a function relating the length of Jonathan's shadow to his distance from the base of the streetlight.
- Determine the rate at which Jonathan's shadow is lengthening at the moment that he is 20 feet from the base of the light.

**552.** A spherical balloon is inflated with helium at the rate of  $100\pi$  ft<sup>3</sup>/min. How fast is the balloon's radius increasing at the instant the radius is 5 ft? How fast is the surface area increasing?

**553.** On a morning of a day when the sun will pass directly overhead, the shadow of an 80-ft building on level ground is 60 feet long. At the moment in question, the angle  $\theta$  the sun's rays make with the ground is increasing at the rate of  $\frac{3\pi}{2000}$  radian/min. At what rate is the shadow decreasing? (Express your answer in inches per minute.)

**554 (AP, 1970AB).** A right circular cone and a hemisphere have the same base, and the cone is inscribed in the hemisphere. The figure is expanding in such a way that the combined surface area of the hemisphere and its base is increasing at a constant rate of 18 square inches per second. At what rate is the volume of the cone changing at the instant when the radius of the common base is 4 inches?

**555 (AP, 1976AB).** Consider the hyperbola  $3x^2 - y^2 = 23$ .

- A point moves on the hyperbola so that its  $y$ -coordinate is increasing at a constant rate of 4 units per second. How fast is the  $x$ -coordinate changing when  $x = 4$ ?
- For what value of  $k$  will the line  $2x + 9y + k = 0$  be normal to the hyperbola?

---

In the mathematics I can report no deficiency, except that it be that men do not sufficiently understand the excellent use of the pure mathematics, in that they do remedy and cure many defects in the wit and faculties intellectual. For if the wit be too dull, they sharpen it; if too wandering, they fix it; if too inherent in the sense, they abstract it. So that as tennis is a game of no use in itself, but of great use in respect that it maketh a quick eye and a body ready to put itself into all postures; so in the mathematics, that use which is collateral and intervenient is no less worthy than that which is principial and intended. —*Roger Bacon*

## 2.15 Excitement with Derivatives!

FIND  $y'$  FOR EACH OF THE FOLLOWING.

556.  $y = e^{2x}$

562.  $y = 2^{\sin x}$

568.  $y = \ln(\sin x)$

557.  $y = e^{-3x/2}$

563.  $y = 9^{-x}$

569.  $y = (\ln x)^2$

558.  $y = x^2 e^x$

564.  $y = \frac{e^{5x}}{x^2}$

570.  $y = \log_3(1 + x)$

559.  $y = 5e^{2-x}$

565.  $y = \ln(x^2)$

571.  $y = \log_9 \sqrt{x}$

560.  $y = 8^{2x}$

566.  $y = \ln(2 - x^2)$

572.  $y = x \ln x - x$

561.  $y = 3^{x^2}$

567.  $y = \ln(5x + 1)$

573.  $y = \frac{\ln x}{x^2}$

FIND THE DERIVATIVE OF EACH FUNCTION IN SIMPLEST FACTORED FORM.

574.  $g(x) = x^3 e^{2x}$

580.  $D(x) = \ln(\ln x)$

585.  $M(x) = e^{-2x^3}$

575.  $Z(x) = 4e^{4x^2+5}$

581.  $A(x) = \ln(x^2 + x + 1)^2$

586.  $J(x) = \frac{e^x}{x^3}$

576.  $q(x) = \ln(e^x + 1)$

582.  $q(x) = \ln \sqrt[5]{3x - 2}$

587.  $F(x) = x^2 e^{-4 \ln x}$

577.  $f(x) = \frac{e^x - 1}{e^x + 1}$

583.  $A(x) = \frac{\ln x}{x - 2}$

588.  $f(x) = 10^{3x^2-6x}$

578.  $k(x) = \log_3(x^2 + e^x)$

584.  $B(x) = \frac{x - 2}{\ln x}$

589.  $g(x) = 3^{2x} 2^{3x^2}$

579.  $R(x) = \frac{2^x - 1}{5^x}$

USE IMPLICIT DIFFERENTIATION TO FIND  $\frac{dy}{dx}$ .

590.  $2x - 3y + \ln(xy) = 4$

593.  $y = 4 \sin(x - 3y)$

591.  $4x = \ln(x + 3y - 4) + 5$

594.  $2x = 3 \sin y - 2y$

592.  $\ln e^x - \ln y = e^y$

595.  $\cos(x - 2y) = 3y$

FIND  $\frac{dy}{dx}$  IN SIMPLEST FACTORED FORM.

596.  $y = 3x \csc 2x$

601.  $y = \cos^2 3x - \sin^2 3x$

606.  $y = e^{3x} \tan x$

597.  $y = \frac{\cot 5x}{3x^2}$

602.  $y = e^{\sin x}$

607.  $y = e^{1/x^2}$

598.  $y = \sqrt{\cot 5x}$

603.  $y = 3^{\cos x}$

608.  $y = e^{x^2/4}$

599.  $y = 3 \sin 8x \cos 8x$

604.  $y = \log_3(\sin 2x)$

609.  $y = \ln(\sec x + \tan x)$

600.  $y = \frac{\ln x}{\sin x}$

605.  $y = x e^{\ln 3x}$

610.  $y = x e^{\tan x}$

## 2.16 Derivatives of Inverses

FIND THE INVERSE  $f^{-1}$  OF THE FOLLOWING FUNCTIONS  $f$ .

**611.**  $f(x) = \sqrt[3]{x}$

**615.**  $f(x) = e^{2x}$

**612.**  $f(x) = \sqrt{x-1}$

**616.**  $f(x) = \ln(x-3)$

**613.**  $f(x) = \frac{x+2}{3}$

**617.**  $f(x) = 5^{2x-1}$

**618.**  $f(x) = \log_2 x$

**614.**  $f(x) = \frac{1}{x}$

**619.**  $f(x) = \frac{2}{x+5}$

FIND THE DERIVATIVE OF THE INVERSE OF  $F$  AT THE POINT  $x = d$ .

**620.**  $F(x) = x^3 - 4$ ;  $d = 23$

**621.**  $F(x) = \sqrt{2x-5}$ ;  $d = 1$

**622.**  $F(x) = x^2 - 9, x \geq 0$ ;  $d = 7$

**623.**  $F(x) = 4x^5 + 3x^3$ ;  $d = 7$

**624.**  $F(x) = 2x^2 + 10x + 13, x > -\frac{5}{2}$ ;  $d = 1$

**625.**  $F(x) = \sin x$ ;  $d = \frac{1}{2}$

**626.**  $F(x) = \tan x$ ;  $d = 1$

**627.**  $F(x) = 17x^3$ ;  $d = 17$

**628.**  $F(x) = x + \sin x$ ;  $d = 0$

**629.**  $F(x) = \sqrt[3]{x^2-4}$ ;  $d = \sqrt[3]{5}$

FIND  $y'$  FOR EACH OF THE FOLLOWING.

**630.**  $y = \sec^{-1}(5x)$

**634.**  $y = \tan^{-1}\left(\frac{3}{x}\right)$

**631.**  $y = \cos^{-1}(2x-3)$

**635.**  $y = \arccos\left(\frac{1}{x}\right)$

**632.**  $y = \arctan(2x-3)$

**636.**  $y = 2 \sin^{-1} \sqrt{1-2x^2}$

**633.**  $y = \operatorname{arcsec}(3x^2)$

**637.**  $y = \arcsin(1-x)$

**638.** Find an equation for the line tangent to the graph of  $y = e^x$  and that goes through the origin.

---

An expert is someone who knows some of the worst mistakes that can be made in his subject, and how to avoid them. —*Werner Heisenberg*

## 2.17 Dérivé, Derivado, Ableitung, Derivative

**639.** Suppose that functions  $f(x)$  and  $g(x)$  and their first derivatives have the following values at  $x = 0$  and  $x = 1$ .

$x$	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	-3	$\frac{1}{2}$
1	3	5	$\frac{1}{2}$	-4

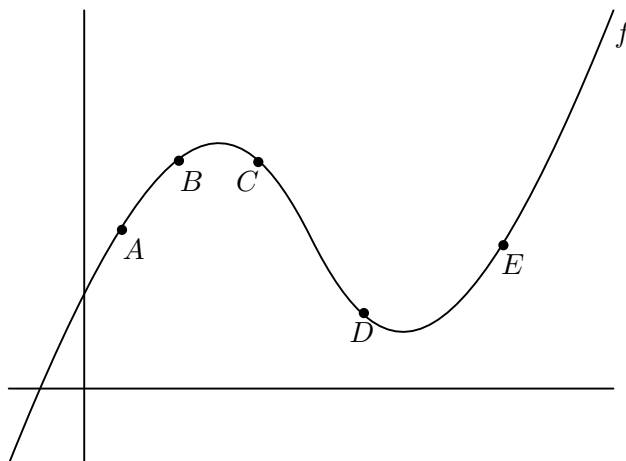
Find the first derivatives of the following combinations at the given value of  $x$ .

- |                                       |                                  |
|---------------------------------------|----------------------------------|
| a) $6f(x) - g(x)$ at $x = 1$          | d) $f(g(x))$ at $x = 0$          |
| b) $f(x)g^2(x)$ at $x = 0$            | e) $g(f(x))$ at $x = 0$          |
| c) $\frac{f(x)}{g(x) + 1}$ at $x = 1$ | f) $(x + f(x))^{3/2}$ at $x = 1$ |
|                                       | g) $f(x + g(x))$ at $x = 0$      |

**640.** If  $x^2 - y^2 = 1$ , find  $\frac{d^2y}{dx^2}$  at the point  $(2, \sqrt{3})$ .

**641.** For what values of  $a$  and  $b$  will  $f(x) = \begin{cases} ax & x < 2 \\ ax^2 - bx + 3 & x \geq 2 \end{cases}$  be differentiable for all values of  $x$ ?

**642.** Use the graph of  $f$  to answer the following.



- Between which two consecutive points is the average rate of change of the function greatest? Least?
- Is the average rate of change of the function between  $A$  and  $B$  greater than or less than the instantaneous rate of change at  $B$ ?
- Sketch a tangent line to the graph between the points  $D$  and  $E$  such that the slope of the tangent is the same as the average rate of the change of the function between  $D$  and  $E$ .
- Give a set of two points for which the average rate of change of the function is approximately equal to another set of two points.



- 643.** The displacement from equilibrium of an object in harmonic motion on the end of a spring is  $y = \frac{1}{3} \cos(12t) - \frac{1}{4} \sin(12t)$  where  $y$  is measured in feet and  $t$  is the time in seconds. Determine the position and velocity of the object when  $t = \frac{\pi}{8}$ .
- 644.** The yield  $Y$ , in millions of cubic feet per acre, for a stand of timber at age  $t$  is  $Y = 6.7e^{-48.1/t}$  where  $t$  is measured in years.
- Find the limiting volume of wood per acre as  $t$  approaches infinity.
  - Find the rate at which the yield is changing when  $t = 20$  years and  $t = 60$  years.
- 645.** Find expressions for the velocity and acceleration of a particle whose position is given by  $x(t) = \sqrt{t} + \sin t$ .
- 646.** The position of a particle is given by  $x(t) = t^3 - 9t^2 + 6t - 3$ . Find the value of the position and velocity of the particle at the time when the acceleration is zero.
- 647.** A ball thrown follows a path described by  $y = x - 0.02x^2$ .
- Sketch a graph of the path.
  - Find the total horizontal distance the ball was thrown.
  - At what  $x$ -value does the ball reach its maximum height?
  - Find an equation that gives the instantaneous rate of change of the height of the ball with respect to the horizontal change. Evaluate this equation at  $x = 0, 10, 25, 30,$  and  $50$ .
  - What is the instantaneous rate of change of the height when the ball reaches its maximum height?
- 648.** A particle moves along the  $x$ -axis so that its position at any time  $t \geq 0$  is  $x(t) = \arctan t$ .
- Prove that the particle is always moving to the right.
  - Prove that the particle is always decelerating.
  - What is the limiting position of the particle as  $t$  approaches infinity?
- 649.** The position at time  $t \geq 0$  of a particle moving along a coordinate line is  $x = 10 \cos(t + \frac{\pi}{4})$ .
- What is the particle's starting position?
  - What are the points farthest to the left and right of the origin reached by the particle?
  - Find the particle's velocity and acceleration at the points in part (b).
  - When does the particle first reach the origin? What are its velocity, speed, and acceleration then?

**2.18 Sample A.P. Problems on Derivatives**

**650.** Let  $f(x) = \begin{cases} x^2 & x \leq 1 \\ 2x & x > 1. \end{cases}$

- a) Find  $f'(x)$  for  $x < 1$ .
- b) Find  $f'(x)$  for  $x > 1$ .
- c) Find  $\lim_{x \rightarrow 1^-} f'(x)$ .
- d) Find  $\lim_{x \rightarrow 1^+} f'(x)$ .
- e) Does  $f'(1)$  exist? Explain.

**651.** Let  $f$  be the function with derivative  $f'(x) = \sin(x^2)$  and  $f(0) = -1$ .

- a) Find the tangent line to  $f$  at  $x = 0$ .
- b) Use your answer to part (a) to approximate the value of  $f$  at  $x = 0.1$ .
- c) Is the actual value of  $f$  at  $x = 0.1$  greater than or less than the approximation from part (b)? Justify your answer.

**652 (1987AB).** Let  $f(x) = \sqrt{1 - \sin x}$ .

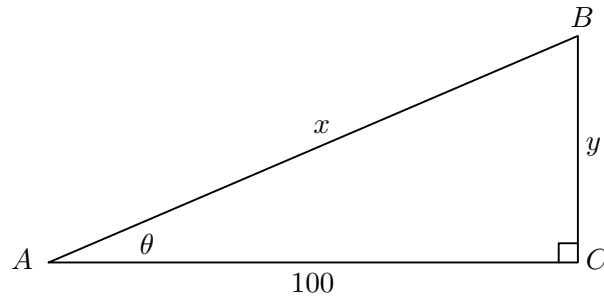
- a) What is the domain of  $f$ ?
- b) Find  $f'(x)$ .
- c) What is the domain of  $f'$ ?
- d) Write an equation for the line tangent to the graph of  $f$  at  $x = 0$ .

**653 (1994AB).** Consider the curve defined by  $x^2 + xy + y^2 = 27$ .

- a) Write an expression for the slope of the curve at any point  $(x, y)$ .
- b) Determine whether the lines tangent to the curve at the  $x$ -intercepts of the curve are parallel. Show the analysis that leads to your conclusion.
- c) Find the points on the curve where the lines tangent to the curve are vertical.

**654 (1994AB).** A circle is inscribed in a square. The circumference of the circle is increasing at a constant rate of 6 inches per second. As the circle expands, the square expands to maintain the condition of tangency.

- a) Find the rate at which the perimeter of the square is increasing. Indicate units of measure.
- b) At the instant when the area of the circle is  $25\pi$  square inches, find the rate of increase in the area enclosed between the circle and the square. Indicate units of measure.



**655 (1988BC).** The figure above represents an observer at point  $A$  watching balloon  $B$  as it rises from point  $C$ . The balloon is rising at a constant rate of 3 meters per second and the observer is 100 meters from point  $C$ .

- Find the rate of change in  $x$  at the instant when  $y = 50$ .
- Find the rate of change in the area of right triangle  $BCA$  at the instant when  $y = 50$ .
- Find the rate of change in  $\theta$  at the instant when  $y = 50$ .

**656 (1990AB).** Let  $f$  be the function given by  $f(x) = \frac{ax + b}{x^2 - c}$  and that has the following properties.

- The graph of  $f$  is symmetric to the  $y$ -axis.
- $\lim_{x \rightarrow 2^+} f(x) = \infty$ .
- $f'(1) = -2$ .

- Determine the values of  $a$ ,  $b$ , and  $c$ .
- Write an equation for each vertical and horizontal asymptote of the graph of  $f$ .
- Sketch the graph of  $f$ .

**657 (1993BC).** Let  $f$  be a function differentiable throughout its domain and that has the following properties.

- $f(x + y) = \frac{f(x) + f(y)}{1 - f(x)f(y)}$  for all real numbers  $x$  and  $y$  in the domain of  $f$ .
- $\lim_{h \rightarrow 0} f(h) = 0$ .
- $\lim_{h \rightarrow 0} \frac{f(h)}{h} = 1$ .

- Show that  $f(0) = 0$ .
- Use the definition of the derivative to show that  $f'(x) = 1 + [f(x)]^2$ . Indicate clearly where properties (i), (ii), and (iii) are used.

---

It has been said that World War One was a chemist's war and that World War Two was a physicist's war. There are those who say that the next World War, if one should occur, will be a mathematician's war. —John H. Curtiss

## 2.19 Multiple-Choice Problems on Derivatives

658. Let  $F(x) = \begin{cases} \frac{x^2 + x}{x} & x \neq 0 \\ 1 & x = 0. \end{cases}$  Which of the following statements are true of  $F$ ?
- I.  $F$  is defined at  $x = 0$ .  
 II.  $\lim_{x \rightarrow 0} F(x)$  exists.  
 III.  $F$  is continuous at  $x = 0$ .
- A) I only      B) II only      C) I, II only      D) II, III only      E) I, II, and III
659. If  $P(x) = (3x + 2)^3$  then the third derivative of  $P$  at  $x = 0$  is
- A) 0      B) 9      C) 54      D) 162      E) 224
660. If  $F(x) = 3x$  then  $F'(5) =$
- A) 0      B)  $\frac{1}{5}$       C)  $-5$       D) 3      E)  $-\frac{1}{5}$
661. The slope of the curve  $y^3 - xy^2 = 4$  at the point where  $y = 2$  is
- A)  $-2$       B)  $\frac{1}{4}$       C)  $-\frac{1}{2}$       D)  $\frac{1}{2}$       E) 2
662. If  $F(x) = x/(x - 1)^2$  then the set of all  $x$  for which  $F(x)$  exists is
- A) all real numbers      B)  $\{x|x \neq -1\}$       C)  $\{x|x \neq \frac{1}{3}\}$       D)  $\{x|x \neq \pm 1\}$       E)  $\{x|x \neq 1\}$
663. If  $\lim_{x \rightarrow b} G(x) = K$ , then which of the following must be true?
- A)  $G'(b)$  exists.  
 B)  $G(x)$  is continuous at  $x = b$ .  
 C)  $G(x)$  is defined at  $x = b$ .  
 D)  $G(b) = K$ .  
 E) None of the above must be true.
664. Which of the following functions are continuous for all real numbers  $x$ ?
- I.  $y = x^{4/3}$       II.  $y = \sqrt[3]{3x - 1}$       III.  $y = \frac{3x - 1}{4x^2 + 5}$
- A) None of these      B) I only      C) II only      D) I, II only      E) I, II, and III
665. The equation of the tangent line to the curve  $y = x^2 - 4x$  at the point where the curve crosses the  $y$ -axis is
- A)  $y = 8x - 4$       B)  $y = -4x$       C)  $y = -4$       D)  $y = 4x$       E)  $y = 4x - 8$
666. The tangent to the curve  $y = 2xe^{-x}$  is horizontal when  $x =$
- A)  $-2$       B) 1      C)  $-1$       D)  $\frac{1}{e}$       E) None of the above

667. If  $y = \ln\left(\frac{e^x}{e^x - 10}\right)$ , then  $\frac{dy}{dx} =$

- A)  $x - \frac{e^x}{e^x - 10}$       B)  $-\frac{1}{e^x}$       C)  $\frac{10}{10 - e^x}$       D) 0      E)  $\frac{e^x - 20}{e^x - 10}$

668. If  $y = \ln(x\sqrt{x^2 + 1})$ , then  $\frac{dy}{dx} =$

- A)  $1 + \frac{x}{x^2 + 1}$       B)  $1 + \frac{1}{x\sqrt{x^2 + 1}}$       C)  $\frac{2x^2 + 1}{x\sqrt{x^2 + 1}}$       D)  $\frac{2x^2 + 1}{x(x^2 + 1)}$       E)  $\frac{x^2 + x + 1}{x(x^2 + 1)}$

669. If  $y = e^{-x} \ln x$  then  $\frac{dy}{dx}$  when  $x = 1$  is

- A) 0      B) Does not exist      C)  $\frac{2}{e}$       D)  $\frac{1}{e}$       E)  $e$

670. The slope of the line tangent to the graph of  $y = \ln x^2$  at  $x = e^2$  is

- A)  $\frac{1}{e^2}$       B)  $\frac{2}{e^2}$       C)  $\frac{4}{e^2}$       D)  $\frac{1}{e^4}$       E)  $\frac{4}{e^4}$

671. If  $y = \ln(x^2 + y^2)$  then the value of  $\frac{dy}{dx}$  at  $(1, 0)$  is

- A) 0      B) -1      C) 1      D) 2      E) undefined

672. If  $z = \frac{3w}{\cos w}$ , then  $\frac{dz}{dw} =$

- A)  $-\frac{3}{\sin w}$       B)  $\frac{3 \cos w - 3w \sin w}{\cos^2 w}$       C)  $\frac{3}{\sin w}$   
 D)  $\frac{3 \cos w + 3w \sin w}{\cos^2 w}$       E) None of the above

673. Find the derivative of  $y = \frac{1}{2 \sin 2x}$ .

- A)  $-\csc 2x \cot 2x$       B)  $-\csc^2 2x$       C)  $-4 \csc 2x \cot 2x$       D)  $\frac{\cos 2x}{2\sqrt{\sin 2x}}$       E)  $4 \sec 2x$

674. If  $y = \sec^2 \sqrt{x}$  then  $\frac{dy}{dx} =$

- A)  $\frac{\sec \sqrt{x} \tan \sqrt{x}}{\sqrt{x}}$       B)  $\frac{\tan \sqrt{x}}{\sqrt{x}}$       C)  $2 \sec \sqrt{x} \tan^2 \sqrt{x}$   
 D)  $\frac{\sec^2 \sqrt{x} \tan \sqrt{x}}{\sqrt{x}}$       E)  $2 \sec^2 \sqrt{x} \tan \sqrt{x}$

675. If  $y = \sin 11x \cos 11x$ , then the derivative of  $y$  is

- A)  $11 \cos 11x$       B)  $11 \cos 22x$       C)  $\sin^2 11x - \cos^2 11x$   
 D)  $-121 \sin^2 11x$       E)  $-121 \sin 11x \cos 11x$

**A.P. Calculus Test Two**  
**Section One**  
**Multiple-Choice**  
**Calculators Allowed**  
Time—45 minutes  
Number of Questions—15

The scoring for this section is determined by the formula

$$[C - (0.25 \times I)] \times 1.8$$

where  $C$  is the number of correct responses and  $I$  is the number of incorrect responses. An unanswered question earns zero points. The maximum possible points earned on this section is 27, which represents 50% of the total test score.

*Directions:* Solve each of the following problems, using the available space for scratch work. After examining the form of the choices, decide which is the best of the choices given and fill in the corresponding choice on your answer sheet. Do not spend too much time on any one problem.

Good Luck!

NAME:

1.  $\lim_{x \rightarrow \infty} \frac{5x^2}{3x^2 + 100000x} =$

- A) 0
- B) 0.005
- C) 1
- D) 1.667
- E) does not exist

---

2. Which of the following functions are not differentiable at  $x = \frac{2}{3}$  ?

I.  $f(x) = \sqrt[3]{x-2}$       II.  $g(x) = |3x-2|$       III.  $h(x) = |9x^2-4|$

- A) I only
- B) II only
- C) I and II only
- D) II and III only
- E) I and III only

---

3. If  $y = (\ln x)^3$ , then  $dy/dx =$

- A)  $\frac{3}{x}(\ln x)^2$
- B)  $3(\ln x)^2$
- C)  $3x(\ln x)^2 + (\ln x)^3$
- D)  $3(\ln x + 1)$
- E) None of these

4. If  $F(x) = x \sin x$ , then find  $F'(3\pi/2)$ .

- A) 0
  - B) 1
  - C)  $-1$
  - D)  $3\pi/2$
  - E)  $-3\pi/2$
- 

5. The approximate equation of the tangent line to  $f(x) = \cos^2(3x)$  at  $x = \pi/18$  is

- A)  $y = -2.598x + 1.203$
  - B)  $y = 2.598x - 1.203$
  - C)  $y = -2.598x + 0.575$
  - D)  $y = 2.598x - 0.575$
  - E) None of these
- 

6. The slope of the tangent to the curve  $y^3x + y^2x^2 = 6$  at the point  $(2, 1)$  is

- A)  $-\frac{3}{2}$
- B)  $-1$
- C)  $-\frac{5}{14}$
- D)  $-\frac{3}{14}$
- E) 0



7. Which of the following functions has a derivative at  $x = 0$  ?

- I.  $y = \arcsin(x^2 - 1) - x$
- II.  $y = x|x|$
- III.  $y = \sqrt{x^4}$

- A) I only
- B) II only
- C) III only
- D) II and III only
- E) I, II, and III

---

8. When a wholesale produce market has  $x$  crates of lettuce available on a given day, it charges  $p$  dollars per crate as determined by the supply equation  $px - 20p - 6x + 40 = 0$ . If the daily supply is decreasing at the rate of 8 crates per day, at what rate is the price changing when the supply is 100 crates?

- A) not changing
- B) increasing at \$0.10 per day
- C) decreasing at \$0.10 per day
- D) increasing at \$1.00 per day
- E) decreasing at \$1.00 per day

---

9. Suppose a particle is moving along a coordinate line and its position at time  $t$  is given by  $s(t) = \frac{9t^2}{t^2 + 2}$ . For what value of  $t$  in the interval  $[1, 4]$  is the instantaneous velocity equal to the average velocity?

- A) 2.00
- B) 2.11
- C) 2.22
- D) 2.33
- E) 2.44

10. A tangent line drawn to the graph of  $y = \frac{4x}{1+x^3}$  at the point  $(1, 2)$  forms a right triangle with the coordinate axes. The area of the triangle is

- A) 3
  - B) 3.5
  - C) 4
  - D) 4.5
  - E) 5
- 

11. The function

$$f(x) = \begin{cases} 4 - x^2 & x \leq 1 \\ mx + b & x > 1 \end{cases}$$

is continuous and differentiable for all real numbers. What must be the values of  $m$  and  $b$  ?

- A)  $m = 2, b = 1$
  - B)  $m = 2, b = 5$
  - C)  $m = -2, b = 1$
  - D)  $m = -2, b = 5$
  - E) None of these
- 

12. If  $f(x) = -x^2 + x$ , then which of the following expressions represents  $f'(x)$  ?

- A)  $\lim_{h \rightarrow 0} \frac{(-x^2 + x + h) - (-x^2 + x)}{h}$
- B)  $\lim_{h \rightarrow x} \frac{(-x^2 + x + h) - (-x^2 + x)}{h}$
- C)  $\frac{[-(x+h)^2 + (x+h)] - (-x^2 + x)}{h}$
- D)  $\lim_{h \rightarrow 0} \frac{[-(x+h)^2 + (x+h)] - (-x^2 + x)}{h}$
- E) None of these

13. All the functions below, except one, have the property that  $f(x)$  is equal to its fourth derivative,  $f^{(4)}(x)$ . Which one does not have this property?

- A)  $f(x) = \sin x$
  - B)  $f(x) = \cos x$
  - C)  $f(x) = -5e^x$
  - D)  $f(x) = e^{2x}$
  - E)  $f(x) = e^{-x}$
- 

14. If  $g(t) = \frac{\ln t}{e^t}$ , then  $g'(t) =$

- A)  $\frac{1 - \ln t}{e^t}$
  - B)  $\frac{1 - t \ln t}{e^t}$
  - C)  $\frac{t \ln t - 1}{te^t}$
  - D)  $\frac{1 - t \ln t}{te^t}$
  - E)  $\frac{1 - e^t \ln t}{e^{2t}}$
- 

15. If  $H(x) = x^3 - x^2 + \frac{1}{x}$ , which of the following is  $H''(2)$  ?

- A)  $\frac{31}{4}$
- B)  $\frac{39}{4}$
- C)  $\frac{79}{8}$
- D)  $\frac{81}{8}$
- E)  $\frac{41}{4}$

**A.P. Calculus Test Two**  
**Section Two**  
**Free-Response**  
**No Calculators**  
Time—45 minutes  
Number of Questions—3

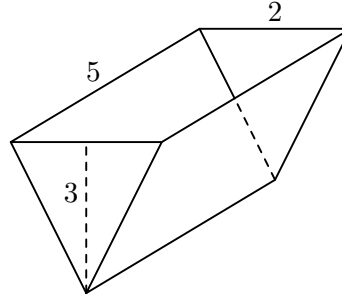
Each of the three questions is worth 9 points. The maximum possible points earned on this section is 27, which represents 50% of the total test score. There is no penalty for guessing.

- **SHOW ALL YOUR WORK.** You will be graded on the methods you use as well as the accuracy of your answers. Correct answers without supporting work may not receive full credit.
- Write all work for each problem in the space provided. Be sure to write clearly and legibly. Erased or crossed out work will not be graded.
- Justifications require that you give mathematical (non-calculator) reasons and that you clearly identify functions, graphs, tables, or other objects that you use.
- Your work must be expressed in mathematical notation rather than calculator syntax. For example,  $y = x^2$  may not be written as  $Y1=X^2$ .
- Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Good Luck!

NAME:

1. Consider the curve defined by the equation  $y + \cos y = x + 1$  for  $0 \leq y \leq 2\pi$ .
    - a) Find  $dy/dx$  in terms of  $y$ .
    - b) Write an equation for each vertical tangent to the curve.
    - c) Find  $\frac{d^2y}{dx^2}$  in terms of  $y$ .
- 



2. The trough shown in the figure above is 5 feet long and its vertical cross sections are inverted isosceles triangles with base 2 feet and height 3 feet. Water is being siphoned out of the trough at the rate of 2 cubic feet per minute. At any time  $t$ , let  $h$  be the depth and  $V$  be the volume of water in the trough.
    - a) Find the volume of water when the trough is full.
    - b) What is the rate of change in  $h$  at the instant when the trough is  $\frac{1}{4}$  full by volume?
    - c) What is the rate of change in the area of the surface of the water at the instant when the trough is  $\frac{1}{4}$  full by volume?
- 
3. Let  $f$  be the function given by  $f(x) = \sqrt{x^4 - 16x^2}$ .
    - a) Find the domain of  $f$ .
    - b) Determine whether  $f$  is an odd or even function.
    - c) Find  $f'(x)$ .
    - d) Find the slope of the line normal to the graph of  $f$  at  $x = 5$ .



## CHAPTER 3

---

# APPLICATIONS of DERIVATIVES

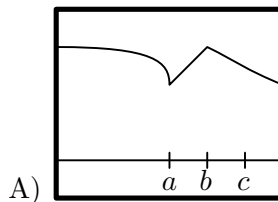
---

### 3.1 The Extreme Value Theorem

IN THE FOUR PROBLEMS BELOW, MATCH THE TABLE WITH THE GRAPH.

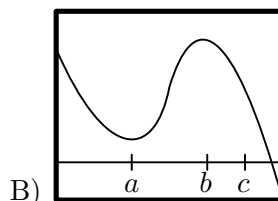
676.

$x$	$f'(x)$
$a$	0
$b$	0
$c$	5



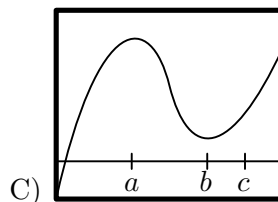
677.

$x$	$f'(x)$
$a$	0
$b$	0
$c$	-5



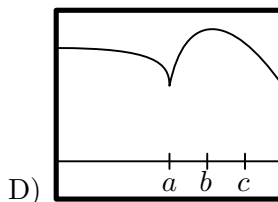
678.

$x$	$f'(x)$
$a$	does not exist
$b$	0
$c$	-2



679.

$x$	$f'(x)$
$a$	does not exist
$b$	does not exist
$c$	-1.7



680. Let  $f(x) = (x - 2)^{2/3}$ .

- Does  $f'(2)$  exist?
- Show that the only local extreme value of  $f$  occurs at  $x = 2$ .
- Does the result in part (b) contradict the Extreme Value Theorem?
- Repeat parts (a) and (b) for  $f(x) = (x - k)^{2/3}$ , replacing 2 with  $k$ .

681. Let  $f(x) = |x^3 - 9x|$ .

- Does  $f'(0)$  exist?
- Does  $f'(3)$  exist?
- Does  $f'(-3)$  exist?
- Determine all extrema of  $f$ .

682. The function  $V(x) = x(10 - 2x)(16 - 2x)$  models the volume of a box. What is the domain of this function? What are the extreme values of  $V$ ?



### 3.2 Rolle to the Extreme with the Mean Value Theorem

IN THE FOLLOWING FOUR PROBLEMS, VERIFY THE THREE CONDITIONS REQUIRED BY ROLLE'S THEOREM AND THEN FIND A SUITABLE NUMBER  $c$  GUARANTEED TO EXIST BY ROLLE'S THEOREM.

**683.**  $f(x) = 2x^2 - 11x + 15$  on  $[\frac{5}{2}, 3]$

**684.**  $g(x) = x^3 + 5x^2 - x - 5$  on  $[-5, -1]$

**685.**  $p(x) = 4x^{4/3} - 6x^{1/3}$  on  $[0, 6]$

**686.**  $k(x) = \frac{x^2 - 4}{x^2 + 4}$  on  $[-2, 2]$

IN THE FOLLOWING EIGHT PROBLEMS, VERIFY THE TWO CONDITIONS REQUIRED BY THE MEAN VALUE THEOREM AND THEN FIND A SUITABLE NUMBER  $c$  GUARANTEED TO EXIST BY THE MEAN VALUE THEOREM.

**687.**  $f(x) = 4x^2 - x - 6$  on  $[1, 3]$

**691.**  $F(x) = x^3$  on  $[1, 3]$

**688.**  $g(x) = \frac{x-1}{x+2}$  on  $[0, 2]$

**692.**  $G(x) = (x-1)^3$  on  $[-1, 2]$

**689.**  $p(x) = 3x^{2/3} - 2x$  on  $[0, 1]$

**693.**  $P(x) = x^2 + 5x$  on  $[0, 2]$

**690.**  $k(x) = x^4 - 3x$  on  $[1, 3]$

**694.**  $H(x) = x^3$  on  $[-1, 3]$

FIND CRITICAL POINTS OF THE FUNCTIONS IN THE FOLLOWING FOUR PROBLEMS.

**695.**  $f(x) = 3x^2 - 5x + 1$

**697.**  $p(x) = \frac{3x-2}{x-4}$

**696.**  $h(x) = x^4 - 2x^2 + 3$

**698.**  $h(x) = 2x^{5/3} - x^{2/3} + 3$

**699.** The function  $f(x) = \begin{cases} x & 0 \leq x < 1 \\ 0 & x = 1 \end{cases}$  is zero at  $x = 0$  and  $x = 1$ , and differentiable on  $(0, 1)$ , but its derivative on  $(0, 1)$  is never zero. Doesn't this contradict Rolle's Theorem?

**700.** A trucker handed in a ticket at a toll booth showing that in 2 hours he had covered 159 miles on a toll road with speed limit 65 mph. The trucker was cited for speeding. Why?

---

I advise my students to listen carefully the moment they decide to take no more mathematics courses. They might be able to hear the sound of closing doors. —James Caballero

### 3.3 The First and Second Derivative Tests

FOR THE FOLLOWING, FIND: A) THE DOMAIN OF EACH FUNCTION, B) THE  $x$ -COORDINATE OF THE LOCAL EXTREMA, AND C) THE INTERVALS WHERE THE FUNCTION IS INCREASING AND/OR DECREASING.

701.  $f(x) = \frac{1}{3}x^3 + \frac{5}{2}x^2 + 6x - 1$

705.  $h(x) = (2 - x)^2(x + 3)^3$

702.  $g(x) = x^3 - 5x^2 - 8x$

706.  $m(x) = 3x\sqrt{5 - x}$

703.  $h(x) = x + \frac{4}{x}$

707.  $f(x) = x^{2/3}(x - 5)^{-1/3}$

704.  $p(x) = \sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}$

708.  $h(x) = \frac{1}{7}x^{7/3} - x^{4/3}$

709. Find the values of  $a$  and  $b$  so that the function  $f(x) = \frac{1}{3}x^3 + ax^2 + bx$  will have a relative extreme at  $(3, 1)$ .

710. Find the values of  $a$ ,  $b$ ,  $c$ , and  $d$  so that the function  $f(x) = ax^3 + bx^2 + cx + d$  will have relative extrema at  $(-1, 1)$  and  $(-2, 4)$ .

IN THE FOLLOWING PROBLEMS, FIND A) THE COORDINATES OF INFLECTION POINTS AND B) THE INTERVALS WHERE THE FUNCTION IS CONCAVE UP AND/OR CONCAVE DOWN.

711.  $g(x) = x^3 - 5x$

712.  $h(x) = 2x^3 - 3x^2 - 8x + 1$

713.  $h(x) = (3x + 2)^3$

714.  $p(x) = \frac{3}{x^2 + 4}$

715.  $f(x) = \begin{cases} x^2 - 3 & x > 3 \\ 15 - x^2 & x \leq 3 \end{cases}$

716.  $p(x) = \begin{cases} 2x^2 & x \geq 0 \\ -2x^2 & x < 0 \end{cases}$

717. Determine the values of  $a$  and  $b$  so that the function  $p(x) = ax^4 + bx^3$  will have a point of inflection at  $(-1, 3)$ .

718. Determine the values of  $a$ ,  $b$ , and  $c$  so that the function  $p(x) = ax^3 + bx^2 + cx$  will have an inflection point at  $(-1, 3)$  and the slope of the tangent at  $(-1, 3)$  will be  $-2$ .

### 3.4 Derivatives and Their Graphs

**719.** The graph of a function  $f$  is given below. Estimate the values of  $f'(x)$  at the following points.

a)  $x = -2$

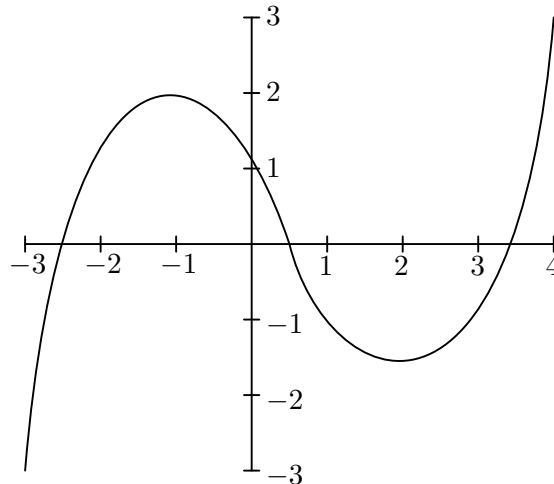
c)  $x = 0$

e)  $x = 2$

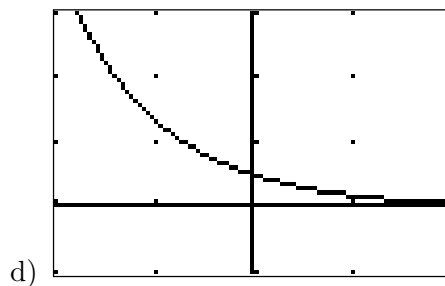
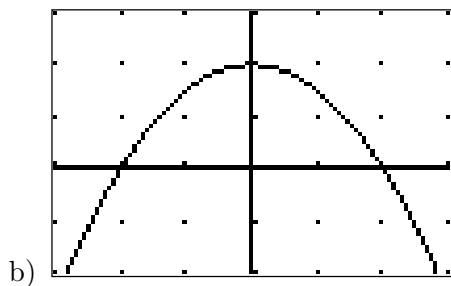
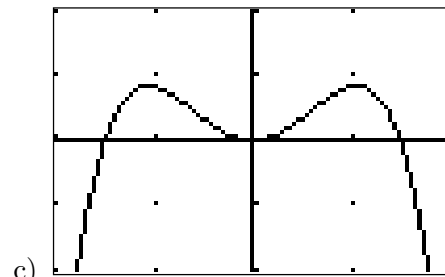
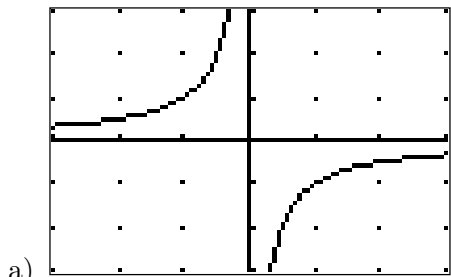
b)  $x = -1$

d)  $x = 1.5$

f)  $x = 3$



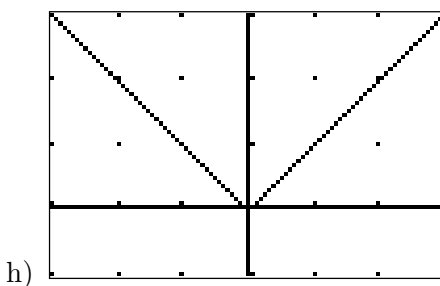
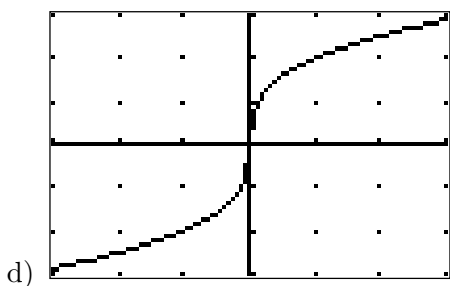
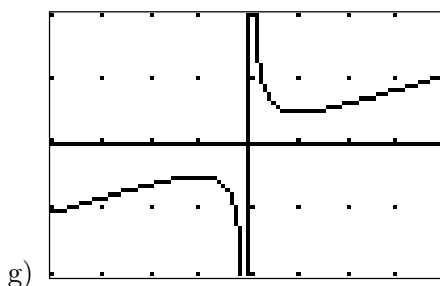
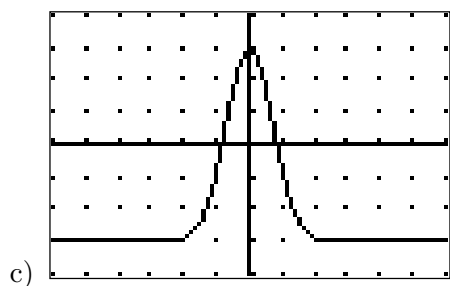
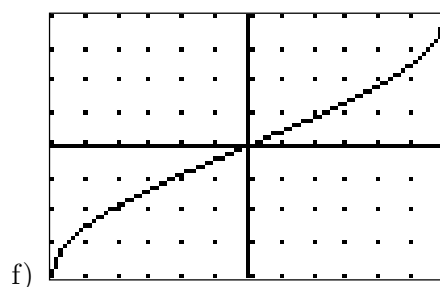
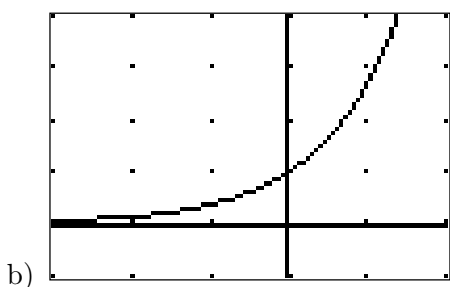
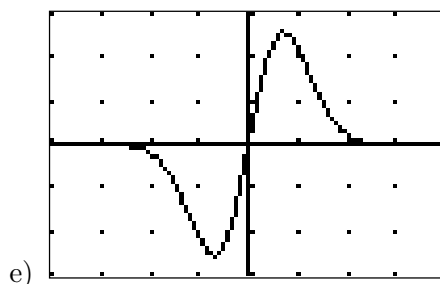
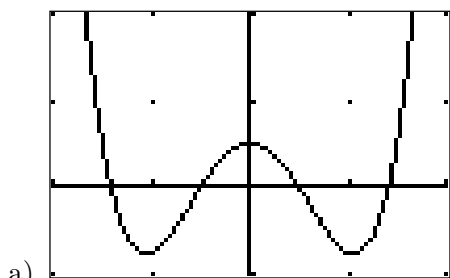
**720.** Sketch the graphs of the derivatives of the four functions shown below.




---

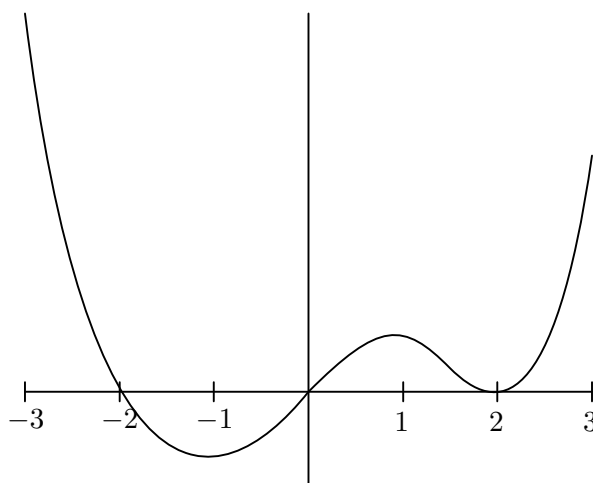
It seems to me that we are all afflicted with an urge and possessed with a longing for the impossible. The reality around us, the three-dimensional world surrounding us, is too common, too dull, too ordinary for us. We hanker after the unnatural or supernatural, that which does not exist, a miracle. —*M. C. Escher*

**721.** The graphs of some functions are given below. Indicate on what intervals the functions are increasing and on what intervals the functions are decreasing, and then sketch the graphs of their derivatives.



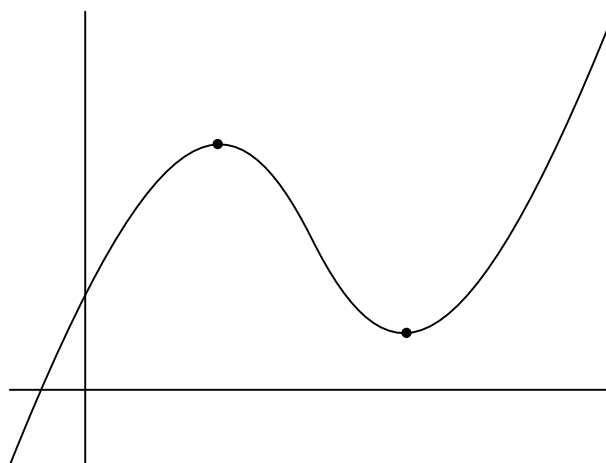
### 3.5 Two Derivative Problems

**722 (AP).** The graph below is the graph of the derivative of a function  $f$ .



- Find where  $f$  is increasing and where it is decreasing.
- Find all local maxima and local minima of  $f$ .
- If  $f(-3) = -2$ , sketch the graph of  $f$ .

**723 (AP).** The graph below is that of a function  $f(x) = ax^3 + bx^2 + cx + d$ , where  $a$ ,  $b$ ,  $c$ , and  $d$  are constants. Show that the  $x$ -coordinates of the two marked points are given by the formula  $x = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}$ .




---

In the fall of 1972 President Nixon announced that the rate of increase of inflation was decreasing. This was the first time a sitting President used the third derivative to advance his case for re-election. —Hugo Rossi

### 3.6 Sketching Functions

FOR THE FOLLOWING SIX PROBLEMS, FIND:

- A) THE DOMAIN
- B) THE ZEROS
- C) THE  $y$ -INTERCEPT
- D) COORDINATES OF LOCAL EXTREMA
- E) INTERVALS WHERE THE FUNCTION INCREASES AND/OR DECREASES
- F) COORDINATES OF INFLECTION POINTS
- G) INTERVALS WHERE THE FUNCTION IS CONCAVE UP AND/OR CONCAVE DOWN, AND THEN
- H) SKETCH THE GRAPH OF THE FUNCTION.

724.  $h(x) = (x - 1)^3(x - 5)$

725.  $f(x) = (x - 2)^{1/3} - 4$

726.  $n(x) = \frac{3x^2}{x^2 - 9}$

727.  $f(x) = x^2e^x$

728.  $j(x) = x \ln x$

729.  $p(x) = \frac{\ln x}{x}$

730. Sketch a graph of a function whose derivative satisfies the properties given in the following table.

$x$	$(-\infty, -1)$	$-1$	$(-1, 1)$	$1$	$(1, 3)$	$3$	$(3, \infty)$
$f'(x)$	positive	0	negative	0	negative	0	positive

731. Suppose  $f$  has a continuous derivative whose values are given in the following table.

$x$	0	1	2	3	4	5	6	7	8	9	10
$f'(x)$	5	2	1	-2	-5	-3	-1	2	3	1	-1

- a) Estimate the  $x$ -coordinates of critical points of  $f$  for  $0 \leq x \leq 10$ .
- b) For each critical point, indicate if it is a local maximum of  $f$ , local minimum, or neither.

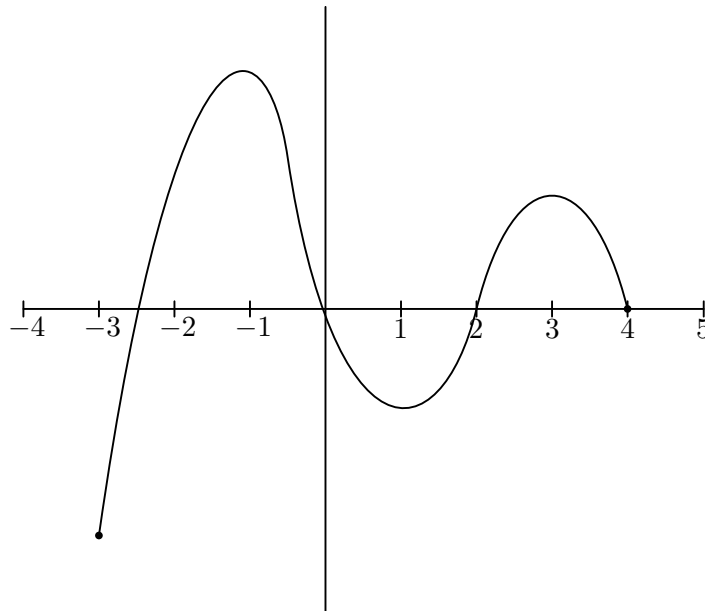
732. Suppose  $f$  is a continuous and differentiable function on the interval  $[0, 1]$  and  $g(x) = f(3x)$ . The table below gives some values of  $f$ .

$x$	0.1	0.2	0.3	0.4	0.5	0.6
$f(x)$	1.01	1.042	1.180	1.298	1.486	1.573

What is the approximate value of  $g'(0.1)$ ?

**733.** The figure below shows the graph of  $g'(x)$ , the derivative of a function  $g$ , with domain  $[-3, 4]$ .

- Determine the values of  $x$  for which  $g$  has a relative minimum and a relative maximum. Justify your answer.
- Determine the values of  $x$  for which  $g$  is concave down and concave up. Justify your answer.
- Based on the information given and the fact that  $g(-3) = 3$  and  $g(4) = 6$ , sketch a possible graph of  $g$ .




---

Bistromathics itself is simply a revolutionary new way of understanding the behavior of numbers. Just as Einstein observed that space was not an absolute but depended on the observer's movement in space, and that time was not an absolute, but depended on the observer's movement in time, so it is now realized that numbers are not absolute, but depend on the observer's movement in restaurants.

The first nonabsolute number is the number of people for whom the table is reserved. This will vary during the course of the first three telephone calls to the restaurant, and then bear no apparent relation to the number of people who actually turn up, or to the number of people who subsequently join them after the show/match/party/gig, or to the number of people who leave when they see who else has turned up.

The second nonabsolute number is the given time of arrival, which is now known to be one of the most bizarre of mathematical concepts, a "recipriversexclusion", a number whose existence can only be defined as being anything other than itself. In other words, the given time of arrival is the one moment of time at which it is impossible that any member of the party will arrive. Recipriversexclusions now play a vital part in many branches of math, including statistics and accountancy and also form the basic equations used to engineer the Somebody Else's Problem field.

The third and most mysterious piece of nonabsoluteness of all lies in the relationship between the number of items on the bill, the cost of each item, the number of people at the table and what they are each prepared to pay for. (The number of people who have actually brought any money is only a subphenomenon of this field.)

—*Douglas Adams, Life, the Universe, and Everything*

### 3.7 Problems of Motion

**734.** A car is moving along Highway 20 according to the given equation, where  $x$  meters is the directed distance of the car from a given point  $P$  at  $t$  hours. Find the values of  $t$  for which the car is moving to the right and when it is moving to the left. Draw a diagram to describe the motion of the car.

a)  $x = 2t^3 + 15t^2 + 36t + 2$

b)  $x = 2t^3 + 9t^2 - 60t - 7$

**735.** A car is moving along Highway 138 according to the given equation, where  $x$  meters is the directed distance of the car from a given point  $P$  at  $t$  hours. Find the values of  $t$  for which the acceleration is zero, and then find the position of the car at this time.

a)  $x = \frac{1}{4}t^4 + \frac{1}{6}t^3 - t^2 + 1$

b)  $x = -3\sqrt{t} - \frac{1}{12\sqrt{t}}$  for  $t > 0$

**736.** A snail moves along the  $x$ -axis so that at time  $t$  its position is given by  $x(t) = 3 \ln(2t - 5)$ , for  $t > \frac{5}{2}$ .

- What is the position and the velocity of the snail at time  $t = 3$ ?
- When is the snail moving to the right, and when is it moving to the left?

**737.** An ant moves along the  $x$ -axis so that at time  $t$  its position is given by  $x(t) = 2 \cos\left(\frac{\pi}{2}t^2\right)$ , for values of  $t$  in the interval  $[-1, 1]$ .

- Find an expression for the velocity of the ant at any given time  $t$ .
- Find an expression for the acceleration at any given time  $t$ .
- Determine the values of  $t$  for which the ant is moving to the right. Justify your answer.
- Determine the values of  $t$  for which the ant changes direction. Justify your answer.

**738.** A particle is moving along the  $x$ -axis so that its position is given by

$$x(t) = \frac{3\pi}{2}t^2 - \sin\left(\frac{3\pi}{2}t^2\right),$$

for  $0 < t \leq 2$ .

- Find an expression for the velocity of the particle at any given time  $t$ .
- Find an expression for the acceleration at any given time  $t$ .
- Find the values of  $t$  for which the particle is at rest.
- Find the position of the particle at the time(s) found in part c).

---

Thus metaphysics and mathematics are, among all the sciences that belong to reason, those in which imagination has the greatest role. I beg pardon of those delicate spirits who are detractors of mathematics for saying this .... The imagination in a mathematician who creates makes no less difference than in a poet who invents.... Of all the great men of antiquity, Archimedes may be the one who most deserves to be placed beside Homer.  
—*Jean le Rond d'Alembert*



**739.** At time  $t \geq 0$ , the velocity of a body moving along the  $x$ -axis is  $v(t) = t^2 - 4t + 3$ .

- Find the body's acceleration each time the velocity is zero.
- When is the body moving forward? Backward?
- When is the body's velocity increasing? Decreasing?

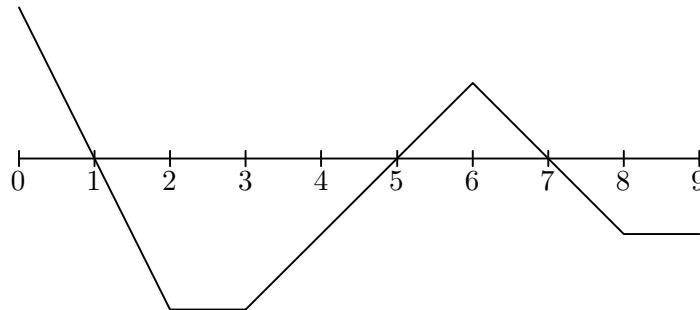
**740.** The position of a ball moving along a straight line is given by  $s(t) = \frac{4}{3}e^{3t} - 8t$ .

- Write an expression for the velocity at any given time  $t$ .
- Write an expression for the acceleration at any given time  $t$ .
- Find the values of  $t$  for which the ball is at rest.
- Find the position of the ball at the time(s) found in part c).

**741.** A racehorse is running a 10 furlong race (1 furlong is 220 yards). As the horse passes each furlong marker,  $F$ , a steward records the time elapsed,  $t$ , since the beginning of the race, as shown in the table below.

$F$	0	1	2	3	4	5	6	7	8	9	10
$t$	0	20	33	46	59	73	86	100	112	124	135

- How long does it take the horse to finish the race?
  - What is the average speed of the horse over the the first 5 furlongs?
  - What is the approximate speed of the horse as it passes the 3-furlong marker?
  - During which portion of the race is the horse running the fastest? Accelerating the fastest?
- 742.** The graph below shows the velocity  $v = f(t)$  of a particle moving on a coordinate line.
- When does the particle move forward? move backward? speed up? slow down?
  - When is the particle's acceleration positive? negative? zero?
  - When does the particle move at its greatest speed?
  - When does the particle stand still for more than an instant?



### 3.8 Maximize or Minimize?

**743.** The famous Kate Lynn Horsefeed is building a box as part of her science project. It is to be built from a rectangular piece of cardboard measuring 25 cm by 40 cm by cutting out a square from each corner and then bending up the sides. Find the size of the corner square which will produce a container that will hold the most.

**744.** Ashley is building a window in the shape of an equilateral triangle whose sides each measure 4 meters. Ashley wants to inscribe a rectangular piece of stained glass in the triangle, so that two of the vertices of the rectangle lie on one of the sides of the triangle. Find the dimensions of the rectangle of maximum area that can be inscribed in the given triangle.

**745.** It has been determined by the brilliant deductive mind of Bruce Wayne that Gotham Highway is located on the line  $y = 2x + 3$ . Determine the point on Gotham Highway closest to the Wayne Foundation Building, which happens to be located at the point  $(1, 2)$ .

**746.** Vaidehi wants to cut a 30-meter piece of iron into two pieces. One of the pieces will be used to build an equilateral triangle, and the other to build a rectangle whose length is three times its width. Where should Vaidehi cut the iron bar if the combined area of the triangle and the rectangle is to be a minimum? How could the combined area of these two figures be a maximum? Justify your answers.

**747.** An open oak wood box with a square base is to be constructed using  $192 \text{ cm}^2$  of oak. If the volume of the box is to be maximized, find its dimensions.

**748.** At the Skywalker moisture farm on the desert planet Tatooine, there are 24 moisture processors, with an average yield per processor of 300 cubits of moisture. Research conducted at Mos Eisley University concludes that when an additional processor is used, the average yield per processor is reduced by 5 cubits. Help Owen and Beru Skywalker find the number of moisture processors that will maximize the number of cubits.

**749.** The fence around Wayne Manor is going to be replaced. No fence will be required on the side lying along Gotham River. If the new wrought iron fence costs \$12 per meter for the side parallel to the river, and \$4 per meter for the other two sides, find the dimensions of the maximum area that can be enclosed by the fence if Bruce Wayne cannot spend more than \$3600.

**750.** The Gotham-Metropolis Highway is a toll road that has averaged 54,000 cars per day over the past five years, with a \$.50 charge per car. A study conducted by the Ray Chulldel Lavet University concludes that for every \$.05 increase in the toll, the number of cars will be reduced by 500. In order to maximize revenue, what toll should the highway charge?

**751.** The range  $R$  of a projectile whose muzzle velocity in meters per second is  $v$ , and whose angle of elevation in radians is  $\theta$ , is given by  $R = (v^2 \sin(2\theta))/g$  where  $g$  is the acceleration of gravity. Which angle of elevation gives the maximum range of the projectile?

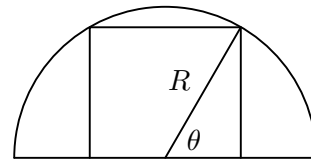
**752.** A piece of wire 100 cm long is to be cut into several pieces and used to construct the skeleton of a box with a square base.

- What is the largest possible volume that such a box can have?
- What is the largest possible surface area?

**753.** In medicine, the reaction  $R(x)$  to a dose  $x$  of a drug is given by  $R(x) = Ax^2(B-x)$ , where  $A > 0$  and  $B > 0$ . The sensitivity  $S(x)$  of the body to a dose of size  $x$  is defined to be  $R'(x)$ . Assume that a negative reaction is a bad thing.

- What seems to be the domain of  $R$ ? What seems to be the physical meaning of the constant  $B$ ? What seems to be the physical meaning of the constant  $A$ ?
- For what value of  $x$  is  $R$  a maximum?
- What is the maximum value of  $R$ ?
- For what value of  $x$  is the sensitivity a minimum?
- Why is it called sensitivity?

**754.** What is the area of the largest rectangle that can be inscribed in a semicircle of radius  $R$  so that one of the sides of the rectangle lies on the diameter of the semicircle?



**755.** An electronics store needs to order a total of 2400 CD players over the course of a year. It will receive them in several shipments, each containing an equal number of CD players. The shipping costs are \$50 for each shipment, plus a yearly fee of \$2 for each CD player in a single shipment. What size should each shipment be in order to minimize yearly shipping costs?

**756.** A rectangle in the first quadrant has one side on the  $y$ -axis, another on the  $x$ -axis, and its upper right-hand vertex on the curve  $y = e^{-x^2}$ . What is the maximum area of the rectangle?

**757.** The positions of two particles on the  $x$ -axis are  $x_1 = \sin t$  and  $x_2 = \sin(t + \frac{\pi}{3})$ .

- At what time(s) in the interval  $[0, 2\pi]$  do the particles meet?
- What is the farthest apart that the particles ever get?
- When in the interval  $[0, 2\pi]$  is the distance between the particles changing the fastest?

**758.** One of the formulas for inventory management says that the average weekly cost of ordering, paying for, and holding merchandise is

$$A(q) = \frac{km}{q} + cm + \frac{hq}{2}$$

where  $q$  is the quantity ordered when things run low,  $k$  is the cost of placing an order (a constant),  $m$  is the number of items sold each week (a constant),  $h$  is the weekly holding cost per item (a constant), and  $c$  is a constant. What is the quantity that will minimize  $A(q)$ ? (The expression you get for your answer is called the *Wilson lot size formula*.)

**759.** The function  $f(x) = \cot x - \sqrt{2} \csc x$  has an absolute maximum value on the interval  $[0, \pi]$ . Find its exact value.

### 3.9 More Tangents and Derivatives

FIND THE TANGENT LINES TO EACH OF THE FOLLOWING AT  $x = 0$ .

**760.**  $\sin x$

**761.**  $\cos x$

**762.**  $\tan x$

**763.**  $e^x$

**764.**  $\ln(1 + x)$

**765.**  $(1 + x)^k$ , for nonzero constant  $k$ .

**766.**  $(1 - x)^k$ , for nonzero constant  $k$ .

**767.** Using the tangent lines found above, approximate the values of  $\sin 0.1$ ;  $\cos 0.1$ ;  $\tan 0.1$ ;  $e^{0.1}$ ;  $\ln(1.1)$ ;  $(1.1)^5$ ; and  $(0.9)^4$ .

**768.** As noted in problems 765 and 766,  $k$  is any nonzero constant. Using the tangent found above, approximate  $\sqrt{1.06}$ ;  $\sqrt[3]{1.06}$ ;  $\frac{1}{1.06}$ ; and  $\frac{1}{(1.06)^2}$ . Then, using your calculator, determine the difference in the approximation compared to the more accurate value given by the calculator.

**769.** Let  $f'(x) = (x - 1)e^{-x}$  be the derivative of a function  $f$ . What are the critical points of  $f$ ? On what intervals is  $f$  increasing or decreasing? At what points, if any, does  $f$  have local extrema?

**770.** Let  $f'(x) = (x - 1)^2(x - 2)$  be the derivative of a function  $f$ . What are the critical points of  $f$ ? On what intervals is  $f$  increasing or decreasing? At what points, if any, does  $f$  have local extrema?

**771.** Let  $f$  be a continuous function on  $[0, 3]$  that has the following signs and values as in the table below.

$x$	0	$0 < x < 1$	1	$1 < x < 2$	2	$2 < x < 3$	3
$f(x)$	0	positive	2	positive	0	negative	-2
$f'(x)$	3	positive	0	negative	does not exist	negative	-3
$f''(x)$	0	negative	-1	negative	does not exist	negative	0

Find the absolute extrema of  $f$  and where they occur; find any points of inflection; and sketch a possible graph of  $f$ .

**772.** A particle moves along the  $x$ -axis as described by  $x(t) = 3t^2 - 2t^3$ . Find the acceleration of the particle at the time when the velocity is a maximum.

**773.** Find the values of  $a$ ,  $b$ ,  $c$ , and  $d$  such that the cubic  $f(x) = ax^3 + bx^2 + cx + d$  has a relative maximum at  $(2, 4)$ , a relative minimum at  $(4, 2)$ , and an inflection point at  $(3, 3)$ .

**774.** Show that the point of inflection of  $f(x) = x(x - 6)^2$  lies midway between the relative extrema of  $f$ .

### 3.10 More Excitement with Derivatives!

**775.** Let  $f(x) = |x| + x$ . Does  $f'(0)$  exist? Explain.

**776.** Determine whether the following functions have a derivative at  $x = 0$ .

a)  $f(x) = x|x|$                       b)  $f(x) = x^2|x|$                       c)  $f(x) = x^3|x|$                       d)  $f(x) = x^4|x|$

**777.** Use the definition of the derivative to find  $g'(1)$ : a)  $g(x) = 2x^2 + 3x$ ; b)  $g(x) = \frac{1}{2x+1}$ .

**778.** Find  $\frac{dy}{dx}$  for each of the following.

a)  $y = 2x^{1/3}$

e)  $y = 25x^{-1} + 12x^{1/2}$

b)  $y = 5x^{11}$

f)  $y = (2x - 5)(3x^4 + 5x + 2)$

c)  $y = x \arctan x$

d)  $y = \frac{1}{2}x^{-3/4}$

g)  $y = \frac{x^2 + 2x - 1}{x^2 - 1}$

**779.** What is the slope of the curve  $y = \frac{t}{t+5}$  at the point  $t = 2$ ? What is the equation of the tangent line at this point?

**780.** What is the slope of the curve  $y = \frac{t^2}{t^2+1}$  at the point  $t = 1$ ? What is the equation of the tangent line at this point?

**781.** Consider a function  $f$  which satisfies the following properties.

i)  $f(x+y) = f(x)f(y)$

ii)  $f(0) \neq 0$

iii)  $f'(0) = 1$

a) Show that  $f(0) = 1$ . *Hint:* Let  $x = y = 0$  in (i).

b) Show that  $f(x) \neq 0$  for all  $x$ . *Hint:* Let  $y = -x$  in (ii).

c) Use the definition of the derivative to show that  $f'(x) = f(x)$  for all real numbers  $x$ .

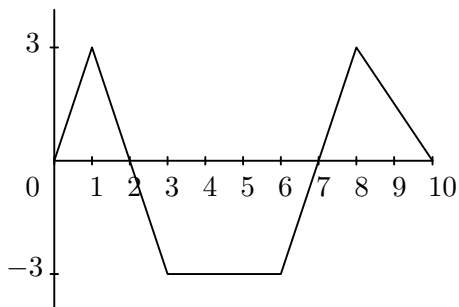
d) There is only one function that satisfies properties (i), (ii), and (iii). Name it.

**782.** If  $\sin x = e^y$ , then find  $\frac{dy}{dx}$  in terms of  $x$ .

**783.** Find  $\lim_{x \rightarrow 0} \frac{e^{3+x} - e^3}{x}$ .

### 3.11 Bodies, Particles, Rockets, Trucks, and Canals

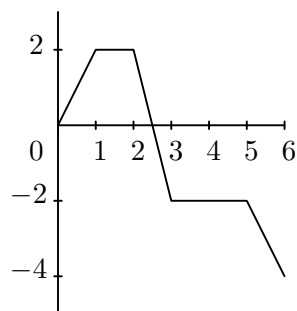
**784.** The graph below shows the velocity  $v(t)$  in meters per second of a body moving along the coordinate line.



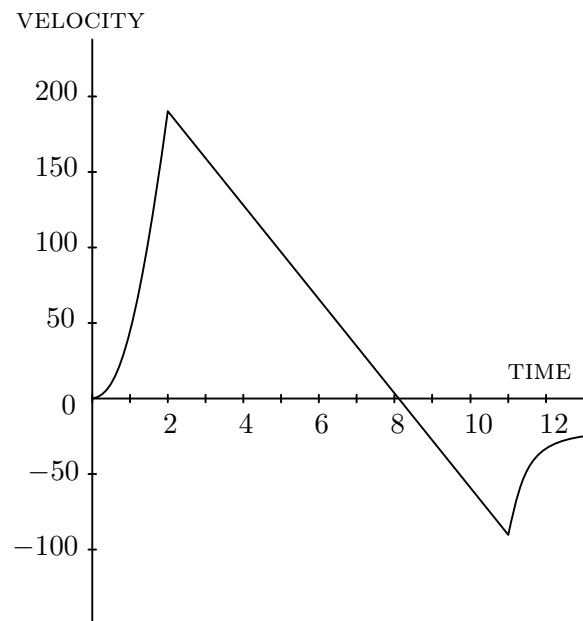
- When does the body reverse direction?
- When is the body moving at a constant speed?
- Graph the body's speed for the interval  $[0, 10]$ .
- Graph the acceleration.

**785.** A particle  $P$  moves along the coordinate line so that the graph at the right is its position  $x(t)$  for time  $t$  in the interval  $[0, 6]$ .

- When is  $P$  moving to the left? Moving to the right? Standing still?
- Graph the particle's velocity and speed.

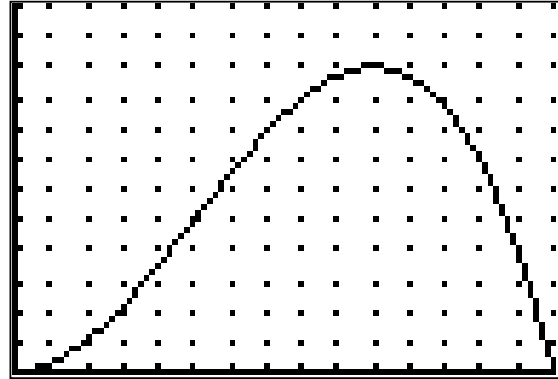


**786.** When a model rocket is launched, the propellant burns for a few seconds, accelerating the rocket upward. After burnout, the rocket coasts upward for a while and then begins to fall. A small explosive charge pops out a parachute shortly after the rocket starts down. The parachute slows the rocket to keep it from breaking when it lands. The figure here shows the velocity from the flight of a model rocket.



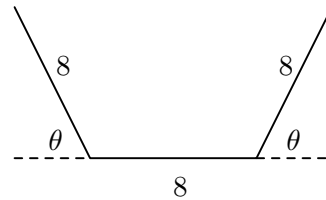
- How fast was the rocket climbing when the engine stopped?
- For how many seconds did the engine burn?
- When did the rocket reach its highest point? What was its velocity then?
- When did the parachute pop out? How fast was the rocket falling then?
- How long did the rocket fall before the parachute opened?
- When was the rocket's acceleration greatest?
- When was the acceleration constant? What was its value then?

**787.** The graph shows the position  $s$  (for  $0 \leq s < 600$ ) of a truck traveling on a highway. The truck starts at  $t = 0$  and returns 15 hours later at  $t = 15$ . (Note: the vertical axis scale is 50, while the horizontal axis scale is 1.)



- a) Graph the truck's velocity and acceleration for  $0 \leq t \leq 15$ .
- b) Suppose  $s = 15t^2 - t^3$ . Graph  $s'$  and  $s''$  on your calculator and compare with the graphs obtained in part (a).

**788.** The cross sections of an irrigation canal are isosceles trapezoids of which three sides are 8 feet long. Determine the angle of elevation  $\theta$  of the sides so that the area of the cross section is a maximum.



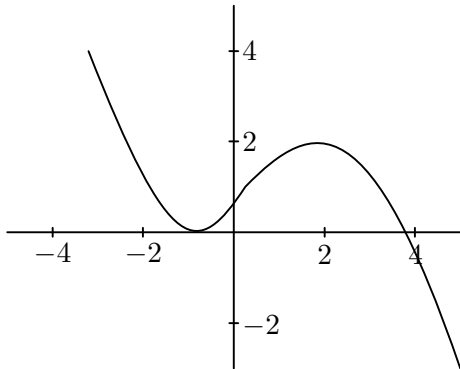
BOTH OF THE FOLLOWING PROBLEMS REFER TO THE GRAPHS BELOW.

**789.** Let  $h(x) = f(x)g(x)$ , where the functions  $f$  and  $g$  are given by the graphs.

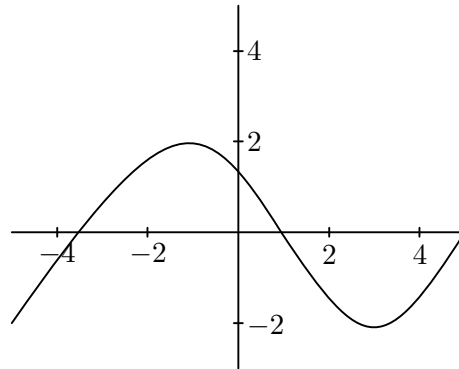
- a) Estimate  $h(-2)$  and  $h(3)$ .
- b) Estimate  $f'(-2)$ ,  $f'(3)$ ,  $g'(-2)$ , and  $g'(3)$ .
- c) Estimate  $h'(-2)$  and  $h'(3)$ .

**790.** Let  $h(x) = f(g(x))$ , where the functions  $f$  and  $g$  are given by the graphs.

- a) Estimate  $h(-2)$  and  $h(3)$ .
- b) Is  $h'(-3)$  positive, negative, or zero? Explain how you know this.
- c) Is  $h'(-1)$  positive, negative, or zero? Explain how you know this.



GRAPH OF  $f$



GRAPH OF  $g$

### 3.12 Even More Excitement with Derivatives!

**791.** Suppose  $f$  and  $g$  are differentiable functions for which:

- i)  $f(0) = 0$  and  $g(0) = 1$ ;
- ii)  $f'(x) = g(x)$  and  $g'(x) = -f(x)$ .

- a) Let  $h(x) = [f(x)]^2 + [g(x)]^2$ . Find  $h'(x)$ , and use this to show that  $[f(x)]^2 + [g(x)]^2 = 1$  for all  $x$ .
- b) Suppose  $F(x)$  and  $G(x)$  are another pair of differentiable functions which satisfy properties (i) and (ii) and let  $k(x) = [F(x) - f(x)]^2 + [G(x) - g(x)]^2$ . Find  $k'(x)$  and use this to discover the relationship between  $f(x)$  and  $F(x)$ , and  $g(x)$  and  $G(x)$ .
- c) Think of a pair of functions  $f$  and  $g$  which satisfy properties (i) and (ii). Can there be any others? Justify your answer.

**792 (AP).** If  $x = \left(\frac{y^2 - 1}{3}\right)^3 - \frac{y^2 - 1}{3}$ , find  $\frac{dy}{dx}$  at the point when  $y = 2$ .

**793 (AP).** Let  $f(x) = x^3 + x$ . If  $h$  is the inverse function of  $f$ , find  $h'(2)$ .

**794 (AP).** For  $-\frac{\pi}{2} < x < \frac{\pi}{2}$ , define  $f(x) = \frac{x + \sin x}{\cos x}$ .

- a) Is  $f$  an even function, an odd function, or neither? Justify your answer.
- b) Find  $f'(x)$ .
- c) Find an equation of the line tangent to the graph of  $f$  at the point where  $x = 0$ .

**795 (AP).** Find all of the following functions that satisfy the equation  $f''(x) = f'(x)$ .

- a)  $f(x) = 2e^x$
- b)  $f(x) = e^{-x}$
- c)  $f(x) = \sin x$
- d)  $f(x) = \ln x$
- e)  $f(x) = e^{2x}$

**796 (AP).** If  $f(x) = e^x$ , which of the following is equal to  $f'(e)$ ?

- A)  $\lim_{\Delta x \rightarrow 0} \frac{e^{x+\Delta x}}{\Delta x}$
- B)  $\lim_{\Delta x \rightarrow 0} \frac{e^{x+\Delta x} - e^e}{\Delta x}$
- C)  $\lim_{\Delta x \rightarrow 0} \frac{e^{x+\Delta x} - e}{\Delta x}$
- D)  $\lim_{\Delta x \rightarrow 0} \frac{e^{x+\Delta x} - 1}{\Delta x}$
- E)  $\lim_{\Delta x \rightarrow 0} e^e \frac{e^{\Delta x} - 1}{\Delta x}$

**797 (AP).** Let  $f(x) = \sin x + \cos x$ . Find  $(f^{-1})'(\sqrt{2})$ .



**798 (AP).** Let  $f$  be a continuous function on  $[-3, 3]$  whose first and second derivatives have the following signs and values.

$x$	$-3 < x < -1$	$-1$	$-1 < x < 0$	$0$	$0 < x < 1$	$1$	$1 < x < 3$
$f'(x)$	positive	0	negative	negative	negative	0	negative
$f''(x)$	negative	negative	negative	0	positive	0	negative

- What are the  $x$ -coordinates of the relative extrema of  $f$  on  $[-3, 3]$ ?
- What are the  $x$ -coordinates of the points of inflection of  $f$  on  $[-3, 3]$ ?
- Sketch a possible graph of  $f$  which satisfies all the given properties.

**799 (AP).** Let  $f$  be a function which is twice differentiable for all real numbers and which satisfies the following properties:

- $f(0) = 1$
- $f'(x) > 0$  for all  $x \neq 0$
- $f$  is concave down for all  $x < 0$  and is concave up for all  $x > 0$ .

Let  $g(x) = f(x^2)$ .

- Sketch a possible graph for  $f$  which takes into account its properties given above.
- Find the  $x$ -coordinates of all relative minimum points of  $g$ . Justify your answer.
- Where is the graph of  $g$  concave up? Justify your answer.
- Use the information obtained in the three previous parts to sketch a possible graph of  $g$ .

FOR THE FOLLOWING SIX PROBLEMS, FIND THE DOMAIN AND COORDINATES OF LOCAL EXTREMA.

**800.**  $P(x) = 10^{x^2-1}$

**801.**  $A(x) = 10^{1-x^2}$

**802.**  $T(x) = 10^{1/(x^2-1)}$

**803.**  $H(x) = e^{3x/(x+1)}$

**804.**  $Y(x) = \log\left(\frac{1}{x}\right)$

**805.**  $A(x) = \log \sqrt{1-x^2}$

FOR THE FOLLOWING THREE PROBLEMS, FIND  $y''$  IN SIMPLEST FACTORED FORM.

**806.**  $y = xe^{-x}$

**807.**  $y = x^2e^x$

**808.**  $y = e^{e^x}$

### 3.13 Sample A.P. Problems on Applications of Derivatives

**809.** Sketch the graph of a continuous function  $f$  with  $f(0) = -1$  and  $f'(x) = \begin{cases} 1 & x < -1 \\ -2 & x > -1. \end{cases}$

**810 (1987BC).** Consider the curve given by the equation  $y^3 + 3x^2y + 13 = 0$ .

- Find  $dy/dx$ .
- Write an equation for the line tangent to the curve at the point  $(2, -1)$ .
- Find the minimum  $y$ -coordinate of any point on the curve. Justify your answer.

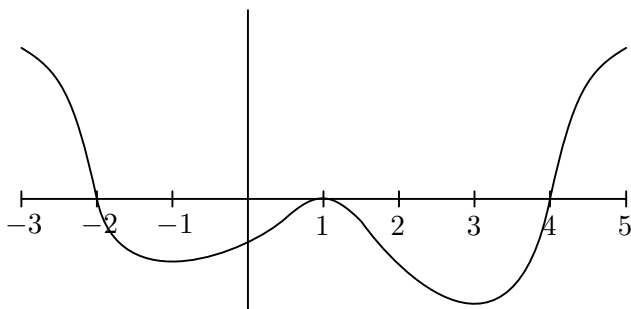
**811 (1990AB).** Let  $f$  be a function defined by  $f(x) = \sin^2 x - \sin x$  for  $0 \leq x \leq \frac{3\pi}{2}$ .

- Find the  $x$ -intercept of the graph of  $f$ .
- Find the intervals on which  $f$  is increasing.
- Find the absolute maximum value and the absolute minimum value of  $f$ . Justify your answer.

**812.** Consider the curve  $y = x^3 + x$ .

- Find the tangents to the curve at the points where the slope is 4.
- What is the smallest slope of the curve?
- At what values  $x$  does the curve have the slope found in part (b)?

**813 (1996AB).** The figure below shows the graph of  $f'$ , the derivative of a function  $f$ . The domain of  $f$  is the set of all real numbers  $x$  such that  $-3 < x < 5$ .



- For what values of  $x$  does  $f$  have a relative maximum? Why?
- For what values of  $x$  does  $f$  have a relative minimum? Why?
- On what intervals is the graph of  $f$  concave upward? Use  $f'$  to justify your answer.
- Suppose that  $f(1) = 0$ . Draw a sketch of  $f$  that shows the general shape of the graph on the open interval  $0 < x < 2$ .

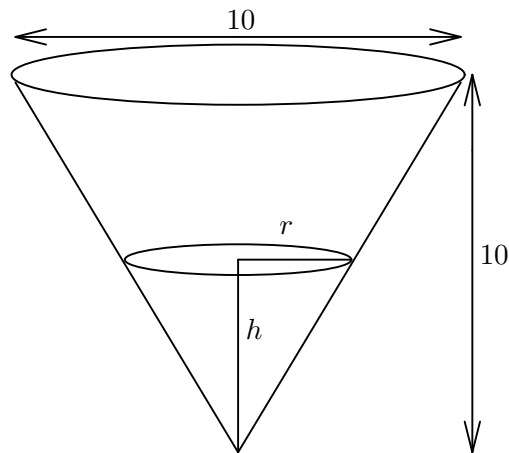
**814 (1992AB).** Let  $f$  be the function given by  $f(x) = \ln \left| \frac{x}{1+x^2} \right|$ .

- Find the domain of  $f$ .
- Determine whether  $f$  is even, odd or neither. Justify your conclusion.
- At what values of  $x$  does  $f$  have a relative maximum or a relative minimum? For each such  $x$ , use the first derivative test to determine whether  $f(x)$  is a relative maximum or a relative minimum.
- Find the range of  $f$ .

**815 (Calculator).** Let  $f(x) = x \ln x$ ,  $a = 0.5$ , and  $b = 3$ .

- Show that  $f$  satisfies the hypotheses of the Mean Value Theorem on the interval  $[a, b]$ .
- Find the value(s) of  $c$  in  $(a, b)$  for which  $f'(c) = \frac{f(b) - f(a)}{b - a}$ .
- Write an equation for the secant line  $AB$  where  $A = (a, f(a))$  and  $B = (b, f(b))$ .
- Write an equation for the tangent line that is parallel to the secant line  $AB$ .

**816 (2002AB).** A container has the shape of an open right circular cone, as shown in the figure below. The height of the container is 10 cm and the diameter of the opening is 10 cm. Water in the container is evaporating so that its depth  $h$  is changing at the constant rate of  $\frac{-3}{10}$  cm/hr.



- Find the volume  $V$  of water in the container when  $h = 5$  cm. Indicate units of measure.
- Find the rate of change of the volume of water in the container, with respect to time, when  $h = 5$  cm. Indicate units of measure.
- Show that the rate of change of the volume of water in the container due to evaporation is directly proportional to the exposed surface area of the water. What is the constant of proportionality?

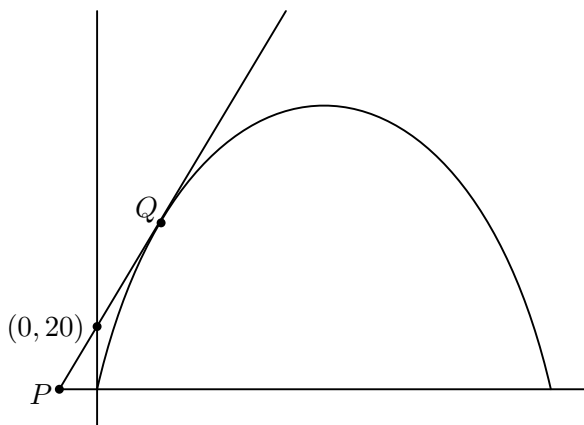
**817 (1992BC).** Let  $f$  be a function defined by  $f(x) = \begin{cases} 2x - x^2 & x \leq 1 \\ x^2 + kx + p & x > 1. \end{cases}$

- For what values of  $k$  and  $p$  will  $f$  be continuous and differentiable at  $x = 1$ ?
- For the value of  $k$  and  $p$  found in part (a), on what interval or intervals is  $f$  increasing?
- Using the values of  $k$  and  $p$  found in part (a), find all points of inflection of the graph of  $f$ . Support your conclusion.

**818 (1989BC).** Consider the function  $f$  defined by  $f(x) = e^x \cos x$  with domain  $[0, 2\pi]$ .

- Find the absolute maximum and minimum values of  $f(x)$ .
- Find intervals on which  $f$  is increasing.
- Find the  $x$ -coordinate of each point of inflection of the graph of  $f$ .

**819 (1996AB).** Line  $\ell$  is tangent to the graph of  $y = x - \frac{x^2}{500}$  at the point  $Q$ , as shown in the figure below.



- Find the  $x$ -coordinate of  $Q$ .
- Write an equation for line  $\ell$ .
- Suppose the graph of  $y$ , where  $x$  and  $y$  are measured in feet, represents a hill. There is a 50-foot tree growing vertically at the top of the hill. Does a spotlight at point  $P$  directed along line  $\ell$  shine on any part of the tree? Show the work that leads to your conclusion.

---

Perhaps the most surprising thing about mathematics is that it is so surprising. The rules which we make up at the beginning seem ordinary and inevitable, but it is impossible to foresee their consequences. These have only been found out by long study, extending over many centuries. Much of our knowledge is due to a comparatively few great mathematicians such as Newton, Euler, Gauss, or Riemann; few careers can have been more satisfying than theirs. They have contributed something to human thought even more lasting than great literature, since it is independent of language. —*E. C. Titchmarsh*

### 3.14 Multiple-Choice Problems on Applications of Derivatives

**820.** The value of  $c$  guaranteed to exist by the Mean Value Theorem for  $V(x) = x^2$  in the interval  $[0, 3]$  is

- A) 1                      B) 2                      C)  $\frac{3}{2}$                       D)  $\frac{1}{2}$                       E) None of these

**821.** If  $P(x)$  is continuous in  $[k, m]$  and differentiable in  $(k, m)$ , then the Mean Value Theorem states that there is a point  $a$  between  $k$  and  $m$  such that

- A)  $\frac{P(k) - P(m)}{m - k} = P'(a)$   
B)  $P'(a)(k - m) = P(k) - P(m)$   
C)  $\frac{m - k}{P(m) - P(k)} = a$   
D)  $\frac{m - k}{P(m) - P(k)} = P'(a)$   
E) None of these

**822.** The Mean Value Theorem does not apply to  $f(x) = |x - 3|$  on  $[1, 4]$  because

- A)  $f(x)$  is not continuous on  $[1, 4]$   
B)  $f(x)$  is not differentiable on  $(1, 4)$   
C)  $f(1) \neq f(4)$   
D)  $f(1) > f(4)$   
E) None of these

**823.** Which of the following function fails to satisfy the conclusion of the Mean Value Theorem on the given interval?

- A)  $3x^{2/3} - 1$ ;  $[1, 2]$   
B)  $|3x - 2|$ ;  $[1, 2]$   
C)  $4x^3 - 2x + 3$ ;  $[0, 2]$   
D)  $\sqrt{x - 2}$ ;  $[3, 6]$   
E) None of these

**824.** If a function  $F$  is differentiable on  $[-4, 4]$ , then which of the following statements is true?

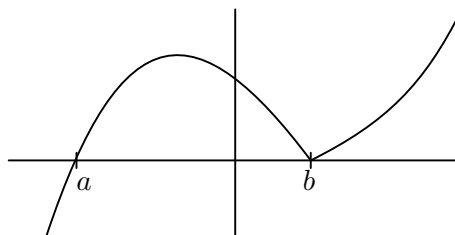
- A)  $F$  is not continuous on  $[-5, 5]$
- B)  $F$  is not differentiable on  $[-5, 5]$
- C)  $F'(c) = 0$  for some  $c$  in the interval  $[-4, 4]$
- D) The conclusion of the Mean Value Theorem applies to  $F$
- E) None of these

**825.** The function  $G(x) = \frac{(x-2)(x-3)}{x-1}$  does not satisfy the hypothesis of Rolle's Theorem on the interval  $[-3, 2]$  because

- A)  $G(-3) = G(2) = 0$
- B)  $G(x)$  is not differentiable on  $[-3, 2]$
- C)  $G(x)$  is not continuous on  $[-3, 2]$
- D)  $G(0) \neq 0$
- E) None of these

**826.** The function  $F$  below satisfies the conclusion of Rolle's Theorem in the interval  $[a, b]$  because

- A)  $F$  is continuous on  $[a, b]$
- B)  $F$  is differentiable on  $(a, b)$
- C)  $F(a) = F(b) = 0$
- D) All three statements A, B and C
- E) None of these



**827.** The intervals for which the function  $F(x) = x^4 - 4x^3 + 4x^2 + 6$  increases are

- A)  $x < 0, 1 < x < 2$
- B) only  $x > 2$
- C)  $0 < x < 1, x > 2$
- D) only  $0 < x < 1$
- E) only  $1 < x < 2$

**828.** If  $Q(x) = (3x + 2)^3$ , then the third derivative of  $Q$  at  $x = 0$  is

- A) 0
- B) 9
- C) 54
- D) 162
- E) 224

**829.** The function  $M(x) = x^4 - 4x^2$  has

- A) one relative minimum and two relative maxima
- B) one relative minimum and one relative maximum
- C) no relative minima and two relative maxima
- D) two relative minima and no relative maxima
- E) two relative minima and one relative maximum

**830.** The total number of all relative extrema of the function  $F$  whose derivative is  $F'(x) = x(x-3)^2(x-1)^4$  is

- A) 0
- B) 1
- C) 2
- D) 3
- E) None of these

**831.** The function  $F(x) = x^{2/3}$  on  $[-8, 8]$  does not satisfy the conditions of the Mean Value Theorem because

- A)  $F(0)$  does not exist
- B)  $F$  is not continuous on  $[-8, 8]$
- C)  $F(1)$  does not exist
- D)  $F$  is not defined for  $x < 0$
- E)  $F'(0)$  does not exist

**832.** If  $c$  is the number defined by Rolle's Theorem, then for  $R(x) = 2x^3 - 6x$  on the interval  $0 \leq x \leq \sqrt{3}$ ,  $c$  must be

- A) 1
- B)  $-1$
- C)  $\pm 1$
- D) 0
- E)  $\sqrt{3}$

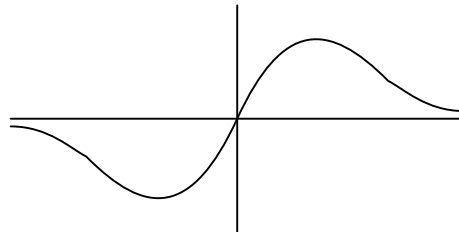
**833.** Find the sum of the values of  $a$  and  $b$  such that  $F(x) = 2ax^2 + bx + 3$  has a relative extremum at  $(1, 2)$ .

- A)  $\frac{3}{2}$
- B)  $\frac{5}{2}$
- C) 1
- D)  $-1$
- E) None of these

**834.** Which of the following statements are true of the graph of  $F(x)$  shown below?

- I. There is a horizontal asymptote at  $y = 0$ .
- II. There are three inflection points.
- III. There are no absolute extrema.

- A) I only
- B) I, II only
- C) I, III only
- D) II, III only
- E) None are true



**A.P. Calculus Test Three**  
**Section One**  
**Multiple-Choice**  
**Calculators Allowed**  
Time—45 minutes  
Number of Questions—15

The scoring for this section is determined by the formula

$$[C - (0.25 \times I)] \times 1.8$$

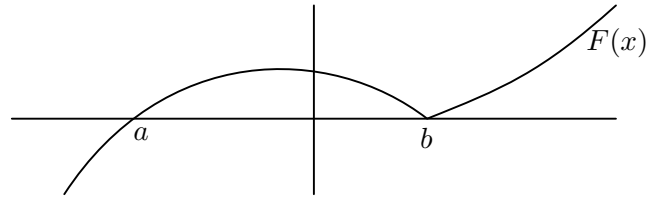
where  $C$  is the number of correct responses and  $I$  is the number of incorrect responses. An unanswered question earns zero points. The maximum possible points earned on this section is 27, which represents 50% of the total test score.

*Directions:* Solve each of the following problems, using the available space for scratch work. After examining the form of the choices, decide which is the best of the choices given and fill in the corresponding choice on your answer sheet. Do not spend too much time on any one problem.

Good Luck!

NAME:





1. The function  $F$  above satisfies the conclusion of Rolle's Theorem in the interval  $[a, b]$  because

- I.  $F$  is continuous.
- II.  $F$  is differentiable on  $(a, b)$ .
- III.  $F(a) = F(b) = 0$ .

- A) I only
- B) II only
- C) I and III only
- D) I, II, and III
- E)  $F$  does not satisfy Rolle's Theorem

---

2. If  $Q(x) = (3x + 2)^3$ , then the third derivative of  $Q$  at  $x = 0$  is

- A) 0
- B) 9
- C) 54
- D) 162
- E) 224

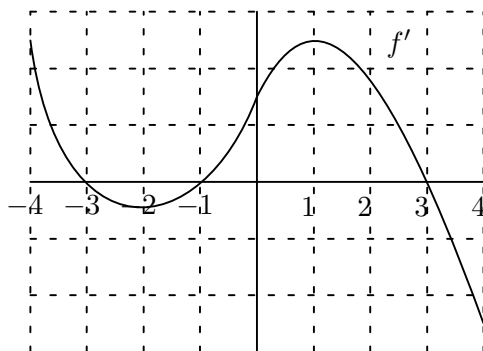
---

3. If a function  $g$  is differentiable on the interval  $[-4, 4]$ , then which of the following statements is true?

- A)  $g$  is not continuous on  $[-5, 5]$ .
- B)  $g$  is not differentiable on  $[-5, 5]$ .
- C)  $g'(c) = 0$  for some  $c$  in  $[-4, 4]$ .
- D) The conclusion of the Mean Value Theorem applies to  $g$ .
- E) None of the above statements are true.

4. The value of  $c$  guaranteed to exist by the Mean Value Theorem for  $f(x) = x^2$  in the interval  $[0, 3]$  is

- A) 1
- B) 2
- C)  $\frac{3}{2}$
- D)  $\frac{1}{2}$
- E) None of these



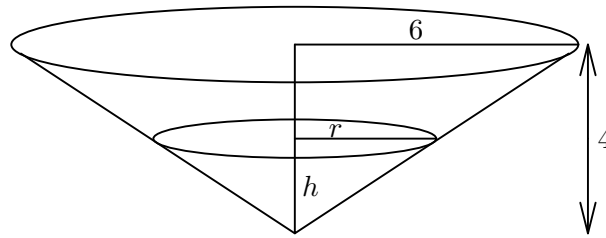
5. The graph of the derivative of a function  $f$  is shown above. Which of the following are true about the original function  $f$ ?

- I.  $f$  is increasing on the interval  $(-2, 1)$ .
- II.  $f$  is continuous at  $x = 0$ .
- III.  $f$  has an inflection point at  $x = -2$ .

- A) I only
- B) II only
- C) III only
- D) II and III only
- E) I, II, and III

6. Two particles move along the  $x$ -axis and their positions at time  $0 \leq t \leq 2\pi$  are given by  $x_1 = \cos t$  and  $x_2 = e^{(t-3)/2} - 0.75$ . For how many values of  $t$  do the two particles have the same velocity?

- A) 0
- B) 1
- C) 2
- D) 3
- E) 4



7. The conical reservoir shown above has diameter 12 feet and height 4 feet. Water is flowing into the reservoir at the constant rate of 10 cubic feet per minute. At the instant when the surface of the water is 2 feet above the vertex, the water level is rising at the rate of

- A) 0.177 ft per min
- B) 0.354 ft per min
- C) 0.531 ft per min
- D) 0.708 ft per min
- E) 0.885 ft per min

8. The position of a particle moving on the  $x$ -axis, starting at time  $t = 0$ , is given by  $x(t) = (t - a)^3(t - b)$ , where  $0 < a < b$ . Which of the following statements are true?

- I. The particle is at a positive position on the  $x$ -axis at time  $t = \frac{a+b}{2}$ .
- II. The particle is at rest at time  $t = a$ .
- III. The particle is moving to the right at time  $t = b$ .

- A) I only
- B) II only
- C) III only
- D) I and II only
- E) II and III only

9. Let the function  $f$  be differentiable on the interval  $[0, 2.5]$  and define  $g$  by  $g(x) = f(f(x))$ . Use the table below to estimate  $g'(1)$ .

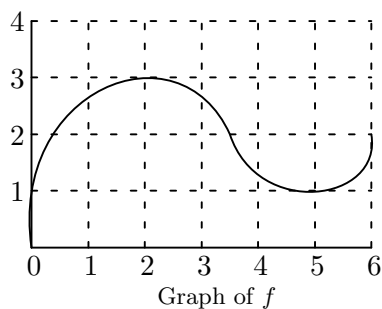
$x$	0.0	0.5	1.0	1.5	2.0	2.5
$f(x)$	1.7	1.8	2.0	2.4	3.1	4.4

- A) 0.8
- B) 1.2
- C) 1.6
- D) 2.0
- E) 2.4

10. Which of the following are true about a particle that starts at  $t = 0$  and moves along a number line if its position at time  $t$  is given by  $s(t) = (t - 2)^3(t - 6)$ ?

- I. The particle is moving to the right for  $t > 5$ .
- II. The particle is at rest at  $t = 2$  and  $t = 6$ .
- III. The particle changes direction at  $t = 2$ .

- A) I only
- B) II only
- C) III only
- D) I and III only
- E) None are true.



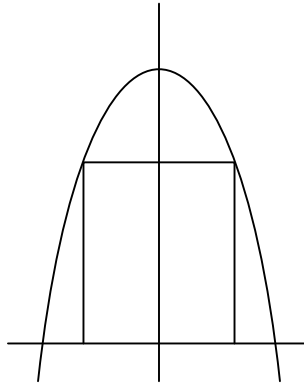
11. The graph of the function  $f$  is shown above. Which of the following statements are true?

- I.  $\lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} = f'(5)$ .
- II.  $\frac{f(5) - f(2)}{5 - 2} = \frac{2}{3}$ .
- III.  $f''(1) \leq f''(5)$ .

- A) I and II only
- B) I and III only
- C) II and III only
- D) I, II, and III
- E) None of these

12. If  $x^2 - y^2 = 25$ , then  $\frac{d^2y}{dx^2} =$

- A)  $-\frac{x}{y}$
- B)  $\frac{5}{y^2}$
- C)  $-\frac{x^2}{y^3}$
- D)  $-\frac{25}{y^3}$
- E)  $\frac{4}{y^3}$



**13.** A rectangle with one side on the  $x$ -axis has its upper vertices on the graph of  $y = 4 - x^2$ , as shown in the figure above. What is the maximum area of the rectangle?

- A) 1.155
- B) 1.855
- C) 3.709
- D) 6.158
- E) 12.316

---

**14.** Let  $f$  be a twice-differentiable function of  $x$  such that, when  $x = c$ ,  $f$  is decreasing, concave up, and has an  $x$ -intercept. Which of the following is true?

- A)  $f(c) < f'(c) < f''(c)$
- B)  $f(c) < f''(c) < f'(c)$
- C)  $f'(c) < f(c) < f''(c)$
- D)  $f'(c) < f''(c) < f(c)$
- E)  $f''(c) < f(c) < f'(c)$

---

**15.** If  $f'(x) = \arctan(x^3 - x)$ , at how many points is the tangent line to the graph of  $f(x)$  parallel to the line  $y = 2x$  ?

- A) None
- B) 1
- C) 2
- D) 3
- E) Infinitely many

**A.P. Calculus Test Three**  
**Section Two**  
**Free-Response**  
**No Calculators**  
Time—45 minutes  
Number of Questions—3

Each of the three questions is worth 9 points. The maximum possible points earned on this section is 27, which represents 50% of the total test score. There is no penalty for guessing.

- **SHOW ALL YOUR WORK.** You will be graded on the methods you use as well as the accuracy of your answers. Correct answers without supporting work may not receive full credit.
- Write all work for each problem in the space provided. Be sure to write clearly and legibly. Erased or crossed out work will not be graded.
- Justifications require that you give mathematical (non-calculator) reasons and that you clearly identify functions, graphs, tables, or other objects that you use.
- Your work must be expressed in mathematical notation rather than calculator syntax. For example,  $y'(2) = 3$  may not be written as `nDeriv(Y1,X,2)=3`.
- Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Good Luck!

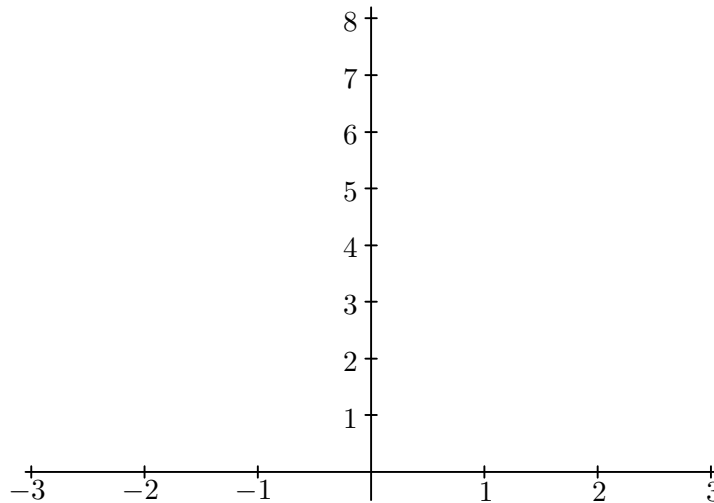
NAME:

1. A particle moves along a line so that at any time  $t$  its position is given by  $x(t) = 2\pi t + \cos 2\pi t$ .
- Find the velocity at time  $t$ .
  - Find the acceleration at time  $t$ .
  - What are all values of  $t$ , for  $0 \leq t \leq 3$ , for which the particle is at rest?
  - What is the maximum velocity?

2. A function  $f$  is continuous on the closed interval  $[-3, 3]$  such that  $f(-3) = 4$  and  $f(3) = 1$ . The function  $f'$  and  $f''$  have the properties given in the table below.

$x$	$-3 < x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$1 < x < 3$
$f'(x)$	positive	fails to exist	negative	0	negative
$f''(x)$	positive	fails to exist	positive	0	negative

- What are the  $x$ -coordinates of all absolute maximum and absolute minimum points of  $f$  on the interval  $[-3, 3]$ ? Justify your answer.
- What are the  $x$ -coordinates of all points of inflection of  $f$  on the interval  $[-3, 3]$ ? Justify your answer.
- On the axes provided, sketch a graph that satisfies the given properties of  $f$ .



3. Let  $f$  be the function given by  $f(x) = x^3 - 5x^2 + 3x + k$ , where  $k$  is a constant.
- On what intervals is  $f$  increasing?
  - On what intervals is the graph of  $f$  concave downward?
  - Find the value of  $k$  for which  $f$  has 11 as its relative minimum.





# CHAPTER 4

---

## **INTEGRALS**

---

## 4.1 The ANTIderivative!

**835.** For each part of this problem you are given two functions,  $f$  and  $g$ . Differentiate both functions. How are the derivatives related? How are  $f$  and  $g$  related? Is it possible for different functions to have the same derivative? What must be true of such functions?

a)  $f(x) = (x - 1)^3$  and  $g(x) = x^3 - 3x^2 + 3x$

b)  $f(x) = \tan^2 x$  and  $g(x) = \sec^2 x$

**836.** Let  $f$  and  $g$  be two differentiable functions such that  $f'(x) = g'(x)$  for all  $x$ . What additional condition from the choices below is necessary in order to conclude that  $f(x) = g(x)$  for all values of  $x$ ?

A)  $f''(x) = g''(x)$  for all  $x$

B)  $f(0) = g(0)$

C)  $f$  and  $g$  are continuous

D) No additional condition will allow you to conclude that  $f(x) = g(x)$

E) No additional condition is required

FIND ANTIDERIVATIVES FOR EACH OF THE FOLLOWING BY CONSIDERING DERIVATIVE RULES IN REVERSE.

**837.**  $6x$

**844.**  $\frac{1}{2x^3}$

**850.**  $4 \sec 3x \tan 3x$

**838.**  $x^7$

**845.**  $x^3 - \frac{1}{x^3}$

**851.**  $\sec \frac{\pi x}{2} \tan \frac{\pi x}{2}$

**839.**  $x^7 - 6x + 8$

**846.**  $\sec^2 x$

**852.**  $x + 1$

**840.**  $-3x^{-4}$

**847.**  $\frac{2}{3} \sec^2 \left(\frac{x}{3}\right)$

**853.**  $3t^2 + \frac{1}{2}t$

**841.**  $x^{-4}$

**848.**  $-\sec^2 \left(\frac{3x}{2}\right)$

**854.**  $\frac{1}{x} - \frac{5}{x^2 + 1}$

**842.**  $x^{-4} + 2x + 3$

**843.**  $-\frac{2}{x^3}$

**849.**  $\sec x \tan x$

**855.**  $\frac{1}{x^2} - x^2 - 3$

**856.** An antiderivative of  $y = e^{x+e^x}$  is

A)  $\frac{e^{x+e^x}}{1+e^x}$

B)  $(1+e^x)e^{x+e^x}$

C)  $e^{1+e^x}$

D)  $e^{x+e^x}$

E)  $e^{e^x}$

---

“Necessity is the mother of invention” is a silly proverb. “Necessity is the mother of futile dodges” is much nearer the truth. —*Alfred North Whitehead*

## 4.2 Derivative Rules Backwards

FIND THE FOLLOWING INDEFINITE INTEGRALS.

$$857. \int (x^3 + 2) dx$$

$$858. \int (x^2 - 2x + 3) dx$$

$$859. \int (x^{3/2} + 2x + 1) dx$$

$$860. \int \left( \sqrt{x} + \frac{1}{2\sqrt{x}} \right) dx$$

$$861. \int \sqrt[3]{x^2} dx$$

$$862. \int \frac{1}{x^3} dx$$

$$863. \int \frac{x^2 + 1}{x^2} dx$$

$$864. \int x^2 \sqrt{x} dx$$

$$865. \int 3 dx$$

$$866. \int (x^2 - \sin x) dx$$

$$867. \int (1 - \csc x \cot x) dx$$

$$868. \int (\sec^2 \theta - \sin \theta) d\theta$$

$$869. \int \sec \theta (\tan \theta - \sec \theta) d\theta$$

$$870. \int \frac{8}{x^{3/5}} dx$$

$$871. \int \frac{-3x}{\sqrt[3]{x^4}} dx$$

$$872. \int 7x^3(3x^4 - 2x) dx$$

$$873. \int \frac{7\sqrt{x} - 3x^2 - 3}{4\sqrt{x}} dx$$

$$874. \int e^x dx$$

$$875. \int 2^x \ln 2 dx$$

$$876. \int 5e^x dx$$

$$877. \int \frac{1}{x^2 + 1} dx$$

$$878. \int \frac{3}{\sqrt{1 - x^2}} dx$$

---

The scientist does not study nature because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful. If nature were not beautiful, it would not be worth knowing, and if nature were not worth knowing, life would not be worth living. Of course I do not here speak of that beauty that strikes the senses, the beauty of qualities and appearances; not that I undervalue such beauty, far from it, but it has nothing to do with science; I mean that profounder beauty which comes from the harmonious order of the parts, and which a pure intelligence can grasp. —*Henri Poincaré*

### 4.3 The Method of Substitution

FIND THE FOLLOWING INDEFINITE INTEGRALS.

$$879. \int -2x\sqrt{9-x^2} \, dx$$

$$880. \int x(4x^2+3)^3 \, dx$$

$$881. \int \frac{x^2}{(1+x^3)^2} \, dx$$

$$882. \int \left(x^2 + \frac{1}{9x^2}\right) \, dx$$

$$883. \int \frac{x^2+3x+7}{\sqrt{x}} \, dx$$

$$884. \int \left(\frac{t^3}{3} + \frac{1}{4t^2}\right) \, dt$$

$$885. \int \sin 2x \, dx$$

$$886. \int \cos 6x \, dx$$

$$887. \int \tan^4 \theta \sec^2 \theta \, d\theta$$

$$888. \int \frac{\sin \theta}{\cos^2 \theta} \, d\theta$$

$$889. \int \cos \frac{\theta}{2} \, d\theta$$

$$890. \int x\sqrt{2x+1} \, dx$$

$$891. \int x^2\sqrt{1-x} \, dx$$

$$892. \int \sqrt{4x-3} \, dx$$

$$893. \int x^4\sqrt{3x^5-4} \, dx$$

$$894. \int \frac{3x^6}{(2x^7-1)^5} \, dx$$

$$895. \int 4x\sqrt{5x-2} \, dx$$

$$896. \int 12x^2 \sin(4x^3) \, dx$$

$$897. \int 4e^x \cos(4e^x) \, dx$$

$$898. \int 3^{3t} \ln 3 \, dt$$

$$899. \int 6^{2x^2-3x} \ln 6 \, dx$$

$$900. \int 2^{5x} \, dx$$

$$901. \int \frac{1}{\sqrt{5x+4}} \, dx$$

$$902. \int 3y\sqrt{7-3y^2} \, dy$$

$$903. \int \cos(3z+4) \, dz$$

$$904. \int \frac{1}{t^2} e^{1/t} \, dt$$

$$905. \int \sec\left(x + \frac{\pi}{2}\right) \tan\left(x + \frac{\pi}{2}\right) \, dx$$

$$906. \int -\csc^2 \theta \sqrt{\cot \theta} \, d\theta$$

$$907. \int \frac{x}{x^2+4} \, dx$$

$$908. \int \frac{1}{\sqrt{1-4x^2}} \, dx$$

$$909. \int \frac{e^x}{1+e^{2x}} \, dx$$

$$910. \int \frac{1}{x} \, dx$$

## 4.4 Using Geometry for Definite Integrals

GRAPH THE INTEGRANDS AND USE GEOMETRY TO EVALUATE THE DEFINITE INTEGRALS.

911.  $\int_{-2}^4 \left(\frac{x}{2} + 3\right) dx$

914.  $\int_{-1}^1 (2 - |x|) dx$

912.  $\int_{-3}^3 \sqrt{9 - x^2} dx$

915.  $\int_0^b x dx$  where  $b > 0$

913.  $\int_{-2}^1 |x| dx$

916.  $\int_a^b 2x dx$  where  $0 < a < b$

917. Suppose  $f$  and  $g$  are continuous and that

$$\int_1^2 f(x) dx = -4, \quad \int_1^5 f(x) dx = 6, \quad \int_1^5 g(x) dx = 8.$$

Evaluate the following definite integrals.

a)  $\int_2^2 g(x) dx$

c)  $\int_1^2 3f(x) dx$

e)  $\int_1^5 [f(x) - g(x)] dx$

b)  $\int_5^1 g(x) dx$

d)  $\int_2^5 f(x) dx$

f)  $\int_1^5 [4f(x) - g(x)] dx$

918. Suppose that  $\int_{-3}^0 g(t) dt = \sqrt{2}$ . Find the following.

a)  $\int_0^{-3} g(t) dt$

b)  $\int_{-3}^0 g(u) du$

c)  $\int_{-3}^0 -g(x) dx$

d)  $\int_{-3}^0 \frac{g(\theta)}{\sqrt{2}} d\theta$

919. A particle moves along the  $x$ -axis so that at any time  $t \geq 0$  its acceleration is given by  $a(t) = 18 - 2t$ . At time  $t = 1$  the velocity of the particle is 36 meters per second and its position is  $x = 21$ .

a) Find the velocity function and the position function for  $t \geq 0$ .

b) What is the position of the particle when it is farthest to the right?

When you feel how depressingly  
Slowly you climb,  
It's well to remember  
That things take time.

—Piet Hein

## 4.5 Some Riemann Sums

**920.** The table shows the velocity of a model train engine moving along a track for 10 seconds. Estimate the distance traveled by the engine using 10 subintervals of length 1 with a) left-hand values and b) right-hand values.

<i>Time (seconds)</i>	0	1	2	3	4	5	6	7	8	9	10
<i>Velocity (in/sec)</i>	0	12	22	10	5	13	11	6	2	6	0

**921.** The table shows the velocity of a vintage sports car accelerating from 0 to 142 miles per hour in 36 seconds (0.01 hours).

<i>hours</i>	0.000	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009	0.010
<i>mph</i>	0	40	62	82	96	108	116	125	132	137	142

- a) Use a Riemann sum to estimate how far the car traveled during the 36 seconds it took to reach 142 mph.
- b) Roughly how many seconds did it take the car to reach the halfway point? About how fast was the car going then?

**922.** Oil is leaking out of a tanker damaged at sea. The damage to the tanker is worsening as evidenced by the increased leakage each hour, recorded in the following table.

<i>Time (hours)</i>	0	1	2	3	4	5	6	7	8
<i>Leakage (gal./hour)</i>	50	70	97	136	190	265	369	516	720

- a) Give an upper and lower estimate of the total quantity of oil that has escaped after 5 hours.
- b) Give an upper and lower estimate of the total quantity of oil that has escaped after 8 hours.
- c) The tanker continues to leak 720 gal/hr after the first 8 hours. If the tanker originally contained 25,000 gallons of oil, approximately how many more hours will elapse in the worst case before all the oil has spilled? In the best case?

**923.** A rectangular swimming pool is 30 ft wide and 50 ft long. The table below shows the depth of the water at 5 ft intervals from one end of the pool to the other. Estimate the volume of water in the pool by computing the average of the left-hand and right-hand Riemann sums.

<i>Position (ft)</i>	0	5	10	15	20	25	30	35	40	45	50
<i>Depth (ft)</i>	6.0	8.2	9.1	9.9	10.5	11.0	11.5	11.9	12.3	12.7	13.0

## 4.6 The MVT and the FTC

FIND  $y'$ , THE DERIVATIVE OF THE FUNCTION  $y$ , FOR EACH OF THE FOLLOWING USING THE FUNDAMENTAL THEOREM OF CALCULUS.

924.  $y = \int_0^x (t + 2) dt$

930.  $y = \int_0^x t \cos t dt$

925.  $y = \int_8^x \sqrt[3]{t} dt$

931.  $y = \int_1^x \frac{t^2}{1+t^2} dt$

926.  $y = \int_{\pi/4}^x \sec^2 t dt$

932.  $y = \int_x^{x+2} (4t + 1) dt$

927.  $y = \int_{-2}^x (t^2 - 2t) dt$

933.  $y = \int_0^{\sin x} \sqrt{t} dt$

928.  $y = \int_{-1}^x \sqrt{t^4 + 1} dt$

934.  $y = \int_0^{x^3} \sin(t^2) dt$

929.  $y = \int_0^x \tan^4 t dt$

935.  $y = \int_0^{3x} \sqrt{1+t^3} dt$

FIND THE AVERAGE VALUE OF EACH OF THE FOLLOWING FUNCTIONS ON THE GIVEN INTERVAL.

936.  $f(x) = x - 2\sqrt{x}$ ;  $[0, 2]$

938.  $f(x) = 2 \sec^2 x$ ;  $[-\frac{\pi}{4}, \frac{\pi}{4}]$

937.  $f(x) = \frac{9}{x^3}$ ;  $[1, 3]$

939.  $f(x) = \cos x$ ;  $[-\frac{\pi}{3}, \frac{\pi}{3}]$

FIND EXACT VALUES FOR EACH OF THE FOLLOWING DEFINITE INTEGRALS.

940.  $\int_0^1 (x^2 + \sqrt{x}) dx$

947.  $\int_0^3 (3x^2 + x - 2) dx$

941.  $\int_0^{\pi/3} 2 \sec^2 x dx$

948.  $\int_1^2 \left( \frac{3}{x^2} - 1 \right) dx$

942.  $\int_{-\pi/2}^{\pi/2} (8y^2 + \sin y) dy$

949.  $\int_{-2}^{-1} \left( u - \frac{1}{u^2} \right) du$

943.  $\int_4^9 \frac{1 - \sqrt{u}}{\sqrt{u}} du$

950.  $\int_{-\pi/3}^{\pi/3} 4 \sec \theta \tan \theta d\theta$

944.  $\int_2^7 3 dx$

951.  $\int_0^2 3^x \ln 3 dx$

945.  $\int_{-1}^8 (x^{1/3} - x) dx$

952.  $\int_0^{\ln 5} e^x dx$

946.  $\int_{-1}^1 (t^2 - 2) dt$

953.  $\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} dx$

### 4.7 The FTC, Graphically

954. Use the function  $f$  in the figure below and the function  $g$  defined by  $g(x) = \int_0^x f(t) dt$ .

a) Complete the table.

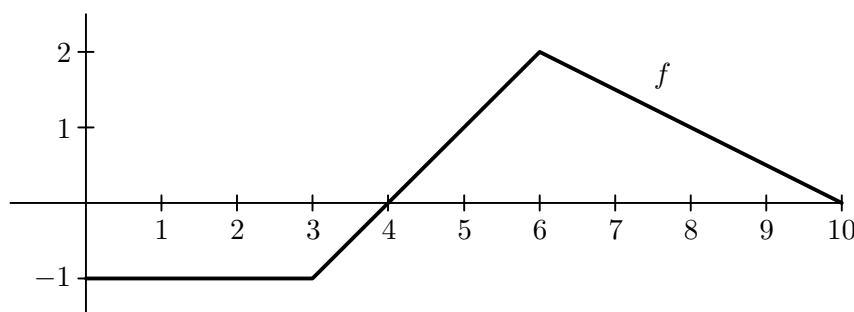
$x$		0		1		2		3		4		5		6		7		8		9		10	
$g(x)$																							

b) Plot the points from the table in part (a).

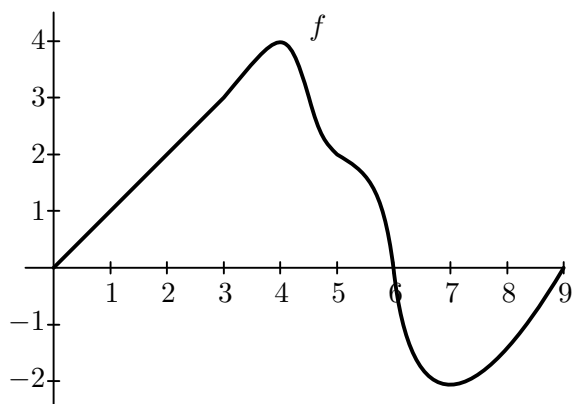
c) Where does  $g$  have its minimum? Explain.

d) Which four consecutive points are collinear? Explain.

e) Between which two consecutive points does  $g$  increase at the greatest rate? Explain.



955. Suppose  $f$  is the differentiable function shown in the accompanying graph and that the position at time  $t$  (in seconds) of a particle moving along the coordinate axis is  $s(t) = \int_0^t f(x) dx$  meters. Use the graph to answer the following questions. Justify your answers.



a) What is the particle's velocity at time  $t = 5$ ?

b) Is the acceleration of the particle at time  $t = 5$  positive or negative?

c) What is the particle's position at  $t = 3$ ?

d) At what time during the first 9 seconds does  $s$  have its largest value?

e) Approximately when is the acceleration zero?

f) When is the particle moving toward the origin? Away from the origin?

g) On which side of the origin does the particle lie at time  $t = 9$ ?



## 4.8 Definite and Indefinite Integrals

FIND THE FOLLOWING INDEFINITE INTEGRALS.

956.  $\int (x^2 - 1)^2 dx$

961.  $\int \frac{z^3 - 2z^2 - 5}{z^2} dz$

957.  $\int \frac{1}{2} \cos 5x dx$

962.  $\int (x^2 + 14x + 49)^{35} dx$

958.  $\int 2^{5w} dw$

963.  $\int e^x (e^x - 1)^7 dx$

959.  $\int \sin(5\theta) \cos(5\theta) d\theta$

964.  $\int [\sin(5\theta) + 1]^4 \cos(5\theta) d\theta$

960.  $\int \frac{4x}{(4x^2 - 1)^5} dx$

965.  $\int 2^{\log_2 7x} dx$

FIND EXACT VALUES FOR THE FOLLOWING DEFINITE INTEGRALS.

966.  $\int_{-1}^1 x(x^2 + 1)^3 dx$

972.  $\int_1^2 (x - 1)\sqrt{2 - x} dx$

967.  $\int_0^1 x\sqrt{1 - x^2} dx$

973.  $\int_0^4 \frac{x}{\sqrt{2x + 1}} dx$

968.  $\int_0^4 \frac{1}{\sqrt{2x + 1}} dx$

974.  $\int_0^{\pi/2} \cos\left(\frac{2x}{3}\right) dx$

969.  $\int_0^2 \frac{x}{\sqrt{1 + 2x^2}} dx$

975.  $\int_{\pi/3}^{\pi/2} (x + \cos x) dx$

970.  $\int_1^9 \frac{1}{\sqrt{x}(1 + \sqrt{x})^2} dx$

976.  $\int_0^7 x\sqrt[3]{x + 1} dx$

971.  $\int_0^2 x\sqrt[3]{x^2 + 4} dx$

977.  $\int_{-2}^6 x^2\sqrt[3]{x + 2} dx$

FIND THE AREA UNDER THE CURVE OVER THE GIVEN INTERVAL.

978.  $y = 2 \sin x + \sin(2x); [0, \pi]$

979.  $y = \sin x + \cos(2x); [0, \pi]$

980.  $y = \sec^2\left(\frac{x}{2}\right); \left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$

981.  $y = \csc(2x) \cot(2x); \left[\frac{\pi}{12}, \frac{\pi}{4}\right]$

---

No one really understood music unless he was a scientist, her father had declared, and not just any scientist, either, oh, no, only the real ones, the theoreticians, whose language is mathematics. She had not understood mathematics until he had explained to her that it was the symbolic language of relationships. "And relationships," he had told her, "contained the essential meaning of life." —*Pearl S. Buck, The Goddess Abides, Part 1*

## 4.9 Integrals Involving Logarithms and Exponentials

FIND THE FOLLOWING INDEFINITE INTEGRALS.

$$982. \int \frac{1}{x+1} dx$$

$$983. \int \frac{x}{x^2+1} dx$$

$$984. \int \frac{x^2-4}{x} dx$$

$$985. \int \frac{x^2+2x+3}{x^3+3x^2+9x} dx$$

$$986. \int \frac{(\ln x)^2}{x} dx$$

$$987. \int \frac{1}{\sqrt{x+1}} dx$$

$$988. \int \frac{\sqrt{x}}{\sqrt{x}-3} dx$$

$$989. \int \frac{2x}{(x-1)^2} dx$$

$$990. \int \frac{\cos \theta}{\sin \theta} d\theta$$

$$991. \int \csc(2\theta) d\theta$$

$$992. \int \frac{\cos \theta}{1+\sin \theta} d\theta$$

$$993. \int \frac{\sec \theta \tan \theta}{\sec \theta - 1} d\theta$$

$$994. \int 5e^{5x} dx$$

$$995. \int \frac{e^{-x}}{1+e^{-x}} dx$$

$$996. \int e^x \sqrt{1-e^x} dx$$

$$997. \int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx$$

$$998. \int \frac{5-e^x}{e^{2x}} dx$$

$$999. \int e^{\sin(\pi x)} \cos(\pi x) dx$$

$$1000. \int e^{-x} \tan(e^{-x}) dx$$

$$1001. \int 3^x dx$$

$$1002. \int 5^{-x^2} x dx$$

$$1003. \int \frac{3^{2x}}{1+3^{2x}} dx$$

FIND EXACT VALUES FOR EACH OF THE FOLLOWING DEFINITE INTEGRALS.

$$1004. \int_0^4 \frac{5}{3x+1} dx$$

$$1005. \int_{-1}^1 \frac{1}{x+2} dx$$

$$1006. \int_e^{e^2} \frac{1}{x \ln x} dx$$

$$1007. \int_0^2 \frac{x^2-2}{x+1} dx$$

$$1008. \int_{\pi}^{2\pi} \frac{1-\cos \theta}{\theta-\sin \theta} d\theta$$

$$1009. \int_1^5 \frac{x+5}{x} dx$$

$$1010. \int_0^1 e^{-2x} dx$$

$$1011. \int_1^3 \frac{e^{3/x}}{x^2} dx$$

$$1012. \int_{-1}^2 2^x dx$$

$$1013. \int_0^1 \frac{3^{4x}(4 \ln 3)}{3^{4x}+1} dx$$

## 4.10 It Wouldn't Be Called the Fundamental Theorem If It Wasn't Fundamental

IN THE FOLLOWING FOUR PROBLEMS, FIND  $F'(x)$ .

1014.  $F(x) = \int_1^x \frac{1}{t} dt$

1016.  $F(x) = \int_x^{3x} \frac{1}{t} dt$

1015.  $F(x) = \int_0^x \tan t dt$

1017.  $F(x) = \int_1^{x^2} \frac{1}{t} dt$

1018. Let  $f$  be a continuous function with an antiderivative  $F$  on the interval  $[a, b]$ . Let  $c$  be any point in the interval. State whether the following are true or false. If false, then correct the statement or give an example to show why it is false.

a)  $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$

b)  $\int_a^b F(x) dx = f(b) - f(a)$

c)  $\int_a^b f(x) dx \geq 0$

d)  $\int_a^b cf(x) dx = c(F(b) - F(a))$

e)  $\int_a^b f(x) dx = f(m)(b - a)$  for some  $m$  in  $[a, b]$

1019. *An Average Value Investigation, Part 1*

- Find the average values of  $f(x) = x$ ,  $f(x) = x^2$ , and  $f(x) = x^3$  over the interval  $[0, 1]$ .
- From the pattern established in part (a), what is the average value of  $f(x) = x^n$ , for an integer  $n \geq 1$ ?
- What does the answer to part (b) imply about the average value of  $f(x) = x^n$ , as  $n$  gets larger and larger? Can you explain this from the graph of  $f(x) = x^n$ ?

1020. *An Average Value Investigation, Part 2*

- Find the average values of  $f(x) = x$ ,  $f(x) = x^{1/2}$ , and  $f(x) = x^{1/3}$  over the interval  $[0, 1]$ .
- From the pattern established in part (a), what is the average value of  $f(x) = x^{1/n}$ , for an integer  $n \geq 1$ ?
- What does the answer to part (b) imply about the average value of  $f(x) = x^{1/n}$ , as  $n$  gets larger and larger? Can you explain this from the graph of  $f(x) = x^{1/n}$ ?

**1021.** Find the average value of the following.

- a)  $f(x) = x - 2$  on  $[1, 3]$
- b)  $f(x) = x^3 - x$  on  $[-1, 1]$
- c)  $f(x) = \cos x$  on  $[0, \pi]$
- d) What is the relationship between the graphs and intervals that make these so easy?

**1022 (AP).** Suppose that  $5x^3 + 40 = \int_c^x f(t) dt$ .

- a) What is  $f(x)$ ?
- b) Find the value of  $c$ .

**1023.** Let  $G(x) = \int_0^x \sqrt{16 - t^2} dt$ .

- a) Find  $G(0)$ .
- b) Does  $G(2) = G(-2)$ ? Does  $G(2) = -G(-2)$ ?
- c) What is  $G'(2)$ ?
- d) What are  $G(4)$  and  $G(-4)$ ?

**1024.** Marcus is caught speeding. The fine is \$3.00 per minute for each mile per hour above the speed limit. Since he was clocked at speeds as much as 64 mph over a 6-minute period, the judge fines him:

$$(\$3.00)(\text{number of minutes})(\text{mph over } 55) = (\$3.00)(6)(64-55) = \$162.00$$

Marcus believes that the fine is too large since he was going 55 mph at times  $t = 0$  and  $t = 6$  minutes, and was going 64 mph only at  $t = 3$ . He reckons, in fact, that his speed  $v$  is given by  $v = 55 + 6t - t^2$ .

- a) Show that Marcus's equation does give the correct speed at times  $t = 0$ ,  $t = 3$  and  $t = 6$ .
- b) Marcus argues that since his speed varied, the fine should be determined by calculus rather than by arithmetic. What should he propose to the judge as a reasonable fine?

**1025.** If  $F(x) = \int_0^3 t\sqrt{t+9} dt$ , then  $F'(1) = 0$ . Why?

**1026.** Evaluate  $\frac{d}{dx} \int_a^b x^3 dx$  where  $a$  and  $b$  are real numbers.

**1027.** If  $g(x) = \int_0^x \sqrt{u^2 + 2} du$ , what is  $\frac{d^2g}{dx^2}$ ?

**1028.** If  $g(x) = \int_0^{x^2} f(u) du$ , what is  $\frac{dg}{dx}$ ?

## 4.11 Definite and Indefinite Integrals Part 2

FIND EXACT VALUES FOR THE FOLLOWING DEFINITE INTEGRALS.

$$1029. \int_{-2}^4 10 \, dx$$

$$1032. \int_{-1}^1 |x| \, dx$$

$$1035. \int_{-\pi/2}^{\pi} |\cos x| \, dx$$

$$1030. \int_4^{-2} dx$$

$$1033. \int_0^2 |2x - 3| \, dx$$

$$1036. \int_0^3 |x^2 - 4| \, dx$$

$$1031. \int_{-1}^1 \sqrt{1-x^2} \, dx$$

$$1034. \int_0^{3\pi/2} |\sin x| \, dx$$

$$1037. \int_{-2}^2 (5 - |x|) \, dx$$

$$1038. \int_0^4 f(x) \, dx \text{ where } f(x) = \begin{cases} 2 & 0 \leq x < 1 \\ 5 & 1 \leq x < \frac{3}{2} \\ 1 & \frac{3}{2} \leq x < 4 \\ 5 & x = 4 \end{cases}$$

$$1039. \int_0^{10} f(x) \, dx \text{ where } f(x) = \begin{cases} 2x & 0 \leq x < 4 \\ 3 & 4 \leq x < 6 \\ 2x & 6 \leq x < 10 \end{cases}$$

$$1040. \int_{1/2}^5 f(x) \, dx \text{ where } f(x) = \begin{cases} \frac{1}{x} & \frac{1}{2} \leq x \leq 2 \\ x & 2 < x \leq 5 \end{cases}$$

$$1041. \int_0^9 f(x) \, dx \text{ where } f(x) = \begin{cases} x^2 & 0 < x < 2 \\ 4 & 2 \leq x < 5 \\ 9 - x & 5 \leq x < 9 \end{cases}$$

$$1042. \int_0^5 g(x) \, dx \text{ where } g(x) = \begin{cases} 2x^3 - 5x^2 + 3 & 0 < x < 2 \\ 10 + x & 2 \leq x < 3 \\ 20 - x & 3 \leq x < 5 \end{cases}$$

$$1043. \int_0^2 f(x) \, dx \text{ where } f(x) = \begin{cases} x^3 & 0 \leq x < 1 \\ 2 - x & 1 < x \leq 2 \end{cases}$$

1044. Find a curve  $y = f(x)$  with the following properties:

- I.  $\frac{d^2y}{dx^2} = 6x$
- II. Its graph passes through  $(0, 1)$
- III. Its graph has a horizontal tangent at  $(0, 1)$

## 4.12 Regarding Riemann Sums

**1045.** Let  $f(x) = x^2 + x$ . Consider the region bounded by the graph of  $f$ , the  $x$ -axis, and the line  $x = 2$ . Divide the interval  $[0, 2]$  into 8 equal subintervals. Draw a picture to help answer the following.

- a) Obtain a lower estimate for the area of the region by using the left-hand endpoint of each subinterval.
- b) Obtain an upper estimate for the area of the region by using the right-hand endpoint of each subinterval.
- c) Find an approximation for the area that is better than either of the answers obtained in parts (a) and (b).
- d) Without calculating the exact area, determine whether the answer in part (c) is larger or smaller than the exact area. Justify your answer.

**1046.** Let  $f(x) = 4 - x^2$ . Repeat problem 1045 with this new function  $f$ .

**1047.** In order to determine the average temperature for the day, meteorologist Sam Anthuh Alun decides to record the temperature at eight times during the day. She further decides that these recordings do not have to be equally spaced during the day because she does not need to make several readings during those periods when the temperature is not changing much (as well as not wanting to get up in the middle of the night). She decides to make one reading at some time during each of the intervals in the table below.

<i>Time</i>	12AM-5AM	5AM-7AM	7AM-9AM	9AM-1PM
<i>Temp</i>	42°	57°	72°	84°

<i>Time</i>	1PM-4PM	4PM-7PM	7PM-9PM	9PM-12AM
<i>Temp</i>	89°	75°	66°	52°

- a) Using Riemann sums, write a formula for the average temperature for this day.
- b) Calculate the average temperature.

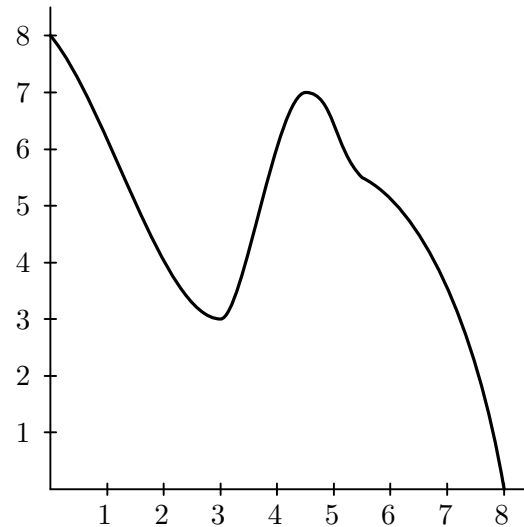
**1048.** Assume the following function  $f$  is a decreasing function on the interval  $[0, 4]$  and that the following is a table showing some function values.

$x$	0	1	1.5	3	4
$f(x)$	4	3	2	1.5	1

Employ a Riemann sum to approximate  $\int_0^4 f(x) dx$ . Use a method so that your approximation will either be less than the value of the definite integral or will be greater than the definite integral. Finally, indicate whether your approximation is less than or greater than the value of the definite integral.

**1049.** Let  $f$  be the function graphed at the right. Which of the following is the best estimate of  $\int_1^6 f(x) dx$ ? Justify your answer.

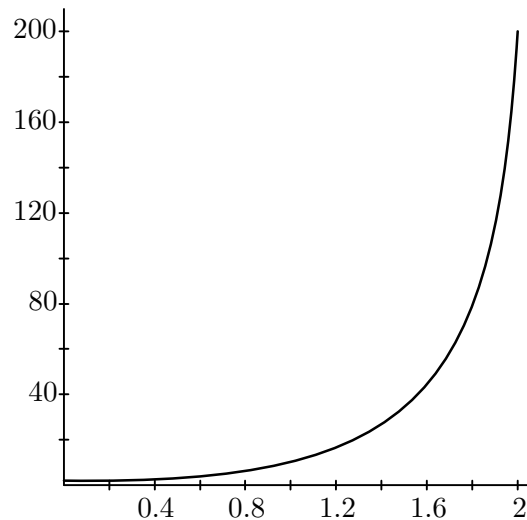
- A)  $-24$
- B)  $9$
- C)  $26$
- D)  $38$



**1050.** The graph of a function  $f$  is given in the figure at right. When asked to estimate  $\int_1^2 f(x) dx$  to five decimal place accuracy, a group of Georgia Southern University calculus students submitted the following answers.

- A)  $-4.57440$
- B)  $4.57440$
- C)  $45.74402$
- D)  $457.44021$

Although one of these responses is correct, the other three are “obviously” incorrect. Using arguments Georgia Southern students would understand, identify the correct answer and explain why each of the others cannot be correct.



**1051.** Consider the following table of values of a continuous function  $f$  at different values of  $x$ .

$x$	1	2	3	4	5	6	7	8	9	10
$f(x)$	0.14	0.21	0.28	0.36	0.44	0.54	0.61	0.70	0.78	0.85

- a) From the data given, find two estimates of  $\int_1^{10} f(x) dx$ .
- b) Obtain a different estimate for the integral by taking an average value of  $f$  over each subinterval.
- c) Do you think that your estimates are too big or too little? Explain.

### 4.13 Definitely Exciting Definite Integrals!

**1052.** Let  $f$  be a continuous function on the interval  $[a, b]$ . State whether the following are true or false. If false, then correct the statement or give an example to show why it is false.

a)  $\frac{d}{dx} \int_a^b f(x) dx = f'(b) - f'(a)$

b)  $\int_a^a f(x) dx = 0$

c)  $\int_a^b f(x) dx = - \int_b^a f(x) dx$

**1053.** Let  $F(x)$  be a continuous function on  $[a, f]$ , where  $a < b < c < d < e < f$ , and

$$\int_a^c F(x) dx = 8, \quad \int_c^e F(x) dx = 5, \quad \int_e^f F(x) dx = -3,$$

$$\int_b^c F(x) dx = 2, \quad \int_d^e F(x) dx = 1.$$

Evaluate the definite integrals below.

a)  $\int_b^e F(x) dx$

f)  $\int_d^c F(x) dx$

b)  $\int_d^e F(x) dx$

g)  $\int_a^e 3F(x) dx$

c)  $\int_a^f F(x) dx$

h)  $\int_c^f -F(x) dx$

d)  $\int_b^d F(x) dx$

i)  $\int_a^b \frac{F(x)}{3} dx + \int_d^f 5F(x) dx$

e)  $\int_b^a F(x) dx$

j)  $\int_f^d F(x) dx - \int_f^b 4F(x) dx$

**1054.** Suppose that  $f$  has a positive derivative for all  $x$  and that  $f(1) = 0$ . Which of the following statements *must* be true of the function  $g(x) = \int_0^x f(t) dt$ ? Justify your answers.

a)  $g$  is a differentiable function of  $x$ .

e)  $g$  has a local minimum at  $x = 1$ .

b)  $g$  is a continuous function of  $x$ .

f) The graph of  $g$  has an inflection point at  $x = 1$ .

c) The graph of  $g$  has a horizontal tangent at  $x = 1$ .

g) The graph of  $dg/dx$  crosses the  $x$ -axis at  $x = 1$ .

d)  $g$  has a local maximum at  $x = 1$ .



## 4.14 How Do I Find the Area Under Thy Curve? Let Me Count the Ways...

IN THE FOLLOWING FOUR PROBLEMS, FIND THE AREA UNDER THE CURVE ON THE INTERVAL  $[a, b]$  BY USING

- A) A RIGHT-HAND RIEMANN SUM ON  $n$  EQUAL SUBINTERVALS;  
 B) A LEFT-HAND RIEMANN SUM ON  $n$  EQUAL SUBINTERVALS;  
 C) 2 TRAPEZOIDS ON EQUAL SUBINTERVALS;  
 D) SIMPSON'S RULE WITH 2 PARABOLAS ON EQUAL SUBINTERVALS; AND  
 E) A DEFINITE INTEGRAL.

**1055.**  $y = 2x + 3$ ;  $[0, 4]$ ;  $n = 4$

**1057.**  $y = 9 - x^2$ ;  $[0, 3]$ ;  $n = 6$

**1056.**  $y = x^2 + 2$ ;  $[1, 3]$ ;  $n = 4$

**1058.**  $y = x^3 + 1$ ;  $[1, 2]$ ;  $n = 2$

FIND THE EXACT AREA OF THE REGION BOUNDED BY THE GIVEN CURVES.

**1059.**  $y = 16 - x^2$ ,  $y = 0$ ,  $x = 0$ ,  $x = -2$

**1062.**  $y = \tan x$ ,  $y = 0$ ,  $x = \frac{\pi}{4}$

**1060.**  $y = x^3 + 4$ ,  $y = 0$ ,  $x = 0$ ,  $x = 1$

**1063.**  $y = \frac{4}{1 + x^2}$ ,  $y = 0$ ,  $x = 0$ ,  $x = 1$

**1061.**  $y = e^{2x}$ ,  $y = 0$ ,  $x = \ln 2$ ,  $x = \ln 3$

FIND THE AVERAGE VALUE OF EACH FUNCTION OVER THE GIVEN INTERVAL.

**1064.**  $F(x) = 2\sqrt{x-1}$ ;  $[1, 2]$

**1066.**  $J(x) = x^n$ ;  $[1, 2]$  for  $n > 1$

**1065.**  $G(x) = e^{-x}$ ;  $[0, 1]$

**1067.**  $W(x) = 3 \cos 3x$ ;  $[0, \frac{\pi}{6}]$

IN THE FOLLOWING PROBLEMS,  $s(t)$  IS POSITION,  $v(t)$  IS VELOCITY, AND  $a(t)$  IS ACCELERATION. FIND BOTH THE NET DISTANCE AND THE TOTAL DISTANCE TRAVELED BY A PARTICLE WITH THE GIVEN POSITION, VELOCITY, OR ACCELERATION FUNCTION.

**1068.**  $v(t) = t^2 - 5t + 6$ , where  $0 \leq t \leq 3$

**1069.**  $s(t) = 3t^3 - t$ , where  $0 \leq t \leq 2$

**1070.**  $a(t) = 2t - 9$ , where  $0 \leq t \leq 3$  and  $v(2) = 13$

**1071.**  $a(t) = -2t + 1$ , where  $0 \leq t \leq 3$  and  $v(0) = 0$

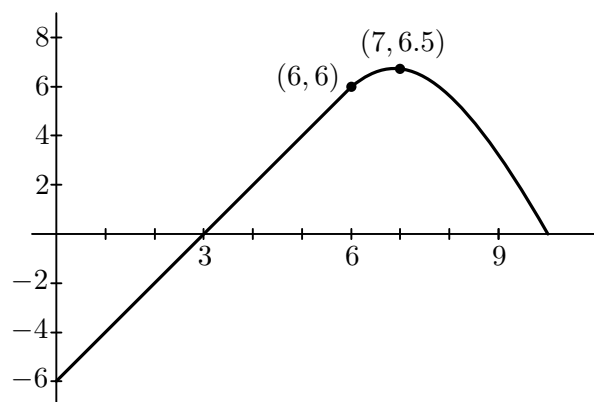
**1072.**  $v(t) = e^{\cos(t/2)} \sin(t/2)$ , where  $0 \leq t \leq 4\pi$

---

The fact is that there are few more “popular” subjects than mathematics. Most people have some appreciation of mathematics, just as most people can enjoy a pleasant tune; and there are probably more people really interested in mathematics than in music. Appearances may suggest the contrary, but there are easy explanations. Music can be used to stimulate mass emotion, while mathematics cannot; and musical incapacity is recognized (no doubt rightly) as mildly discreditable, whereas most people are so frightened of the name of mathematics that they are ready, quite unaffectedly, to exaggerate their own mathematical stupidity. —*G. H. Hardy*

### 4.15 Three Integral Problems

**1073.** Suppose that  $g$  is the differentiable function shown in the accompanying graph and that the position at time  $t$  (in seconds) of a particle moving along a coordinate axis is  $s(t) = \int_0^t g(x) dx$  meters. Use the graph to answer the following questions. Justify your answers.



- What is the particle's velocity at  $t = 3$ ?
- Is the acceleration at time  $t = 3$  positive or negative?
- What is the particle's position at  $t = 3$ ?
- When does the particle pass through the origin?
- When is the acceleration zero?
- When is the particle moving away from the origin? toward the origin?
- On which side of the origin does the particle lie at  $t = 9$ ?

**1074.** Suppose that  $f$  has a negative derivative for all  $x$  and that  $f(1) = 0$ . Which of the following statements must be true of the function  $h(x) = \int_0^x f(t) dt$ ? Justify your answers.

- $h$  is a twice-differentiable function of  $x$ .
- $h$  and  $dh/dx$  are both continuous.
- The graph of  $h$  has a horizontal tangent at  $x = 1$ .
- $h$  has a local maximum at  $x = 1$ .
- $h$  has a local minimum at  $x = 1$ .
- The graph of  $h$  has an inflection point at  $x = 1$ .
- The graph of  $dh/dx$  crosses the  $x$ -axis at  $x = 1$ .

**1075 (Calculator).** *An investigation into the accuracy of the Trapezoid and Simpson's rules*

- Using the Trapezoid rule, approximate the area between the curve  $y = x \sin x$  and the  $x$ -axis from  $x = 0$  to  $x = \pi$ , taking  $n = 4, 8, 20$ , and  $50$  subintervals.
- Repeat part (a) using Simpson's Rule.
- Calculate the value of the definite integral  $\int_0^\pi x \sin x dx$  and compare it to the answers obtained in parts (a) and (b). What does this exercise suggest about the relative accuracy of the trapezoid and Simpson's rules?

## 4.16 Trapezoid and Simpson

APPROXIMATE THE VALUE OF THE FOLLOWING DEFINITE INTEGRALS USING A) THE TRAPEZOID RULE AND B) SIMPSON'S RULE, EACH WITH 4 SUBDIVISIONS. WRITE OUT THE SUM, BUT USE YOUR CALCULATOR TO DO THE ARITHMETIC. YOUR ANSWER MUST BE ACCURATE TO THREE DECIMAL PLACES.

**1076.**  $\int_0^8 \sqrt[3]{x} \, dx$

**1078.**  $\int_0^1 \sqrt{x-x^2} \, dx$

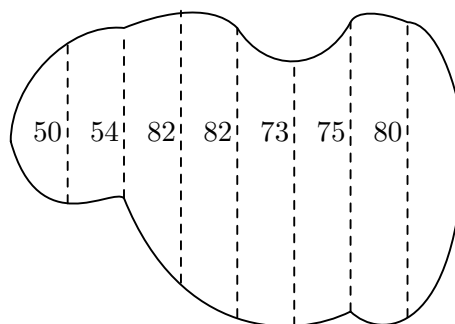
**1080.**  $\int_0^\pi 2^{\sin x} \, dx$

**1077.**  $\int_1^2 \frac{1}{(x+1)^2} \, dx$

**1079.**  $\int_0^4 e^{-x^2} \, dx$

**1081.**  $\int_0^1 \frac{4}{1+x^2} \, dx$

**1082.** To estimate the surface area of a pond, a surveyor takes several measurements, in feet, at 20-foot intervals, as shown in the figure. Estimate the surface area of the pond using a) the trapezoid rule and b) Simpson's rule.



**1083.** The table lists several measurements gathered in an experiment to approximate an unknown continuous function  $y = f(x)$ . Approximate the integral  $\int_0^2 f(x) \, dx$  using a) the trapezoid rule and b) Simpson's rule.

$x$	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
$y$	4.32	4.36	4.58	5.79	6.14	7.25	7.64	8.08	8.14

**1084.** A diesel generator runs continuously, consuming oil at a gradually increasing rate until it must be temporarily shut down to have the filters replaced. Use the trapezoid rule to estimate the amount of oil consumed by the generator during that week.

<i>Day</i>	Sun	Mon	Tue	Wed	Thu	Fri	Sat	Sun
<i>Oil consumption rate (liters/hour)</i>	0.019	0.020	0.021	0.023	0.025	0.028	0.031	0.035

**1085.** An automobile computer gives a digital readout of fuel consumption in gallons per hour. During a trip, a passenger recorded the fuel consumption every five minutes for a full hour of travel. Use the trapezoid rule to approximate the total fuel consumption; then, assuming the automobile covered 60 miles in the hour, find the fuel efficiency (in miles per gallon) for that portion of the trip.

<i>Time</i>	0	5	10	15	20	25	30	35	40	45	50	55	60
<i>Gal/Hr</i>	2.5	2.4	2.3	2.4	2.4	2.5	2.6	2.5	2.4	2.3	2.4	2.4	2.3

## 4.17 Properties of Integrals

**1086.** Suppose that  $f$  is an integrable function and that

$$\int_0^1 f(x) dx = 2, \quad \int_0^2 f(x) dx = 1, \quad \int_2^4 f(x) dx = 7.$$

- Find  $\int_0^4 f(x) dx$ .
- Find  $\int_1^0 f(x) dx$ .
- Find  $\int_1^2 f(x) dx$ .
- Explain why  $f(x)$  must be negative somewhere in the interval  $[1, 2]$ .
- Explain why  $f(x) \geq 3.5$  for at least one value of  $x$  in the interval  $[2, 4]$ .

**1087.** Calculate the exact value of  $\int_{-3}^3 (x+5)\sqrt{9-x^2} dx$ . *Hint:* Consider geometric methods; look at the graphs of  $y = x\sqrt{9-x^2}$  and  $y = \sqrt{9-x^2}$ .

**1088.** Four calculus students disagree as to the value of the integral  $\int_0^\pi \sin^8 x dx$ . Abby says that it is equal to  $\pi$ . Nika says that it is equal to  $35\pi/128$ . Catherine claims it is equal to  $3\pi/90 - 1$ , while Peyton says its equal to  $\pi/2$ . One of them is right. Which one is it? *Hint:* Do not try to evaluate the integral; instead eliminate the three wrong answers.

**1089.** If you were asked to find  $\int_1^2 x^2 e^{x^2} dx$ , you could not do it analytically because you could not find an antiderivative of  $x^2 e^{x^2}$ . However, you should be able to estimate the size of the answer. Which is it?

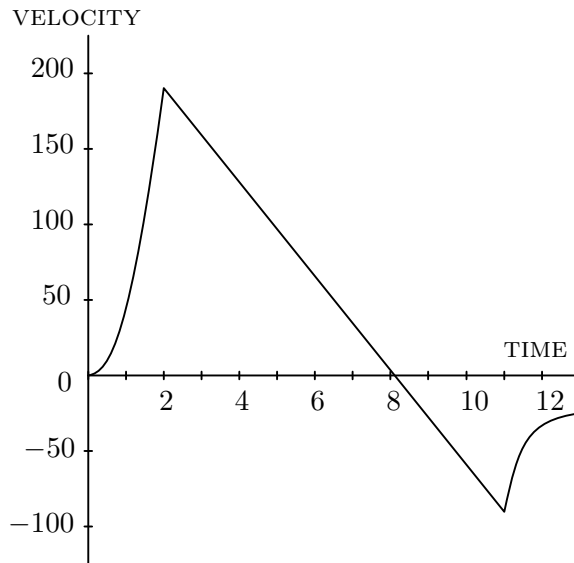
- less than 0
- 0 to 9.999
- 10 to 99.99
- 100 to 999.9
- 1000 to 9999
- over 10,000

---

It is India that gave us the ingenious method of expressing all numbers by means of ten symbols, each symbol receiving a value of position as well as an absolute value; a profound and important idea which appears so simple to us now that we ignore its true merit. But its very simplicity and the great ease which it has lent to computations put our arithmetic in the first rank of useful inventions; and we shall appreciate the grandeur of the achievement the more when we remember that it escaped the genius of Archimedes and Apollonius, two of the greatest men produced by antiquity. —*Pierre Simon Laplace*

### 4.18 Sample A.P. Problems on Integrals

**1090.** The figure shows the graph of the velocity of a model rocket for the first 12 seconds after launch.

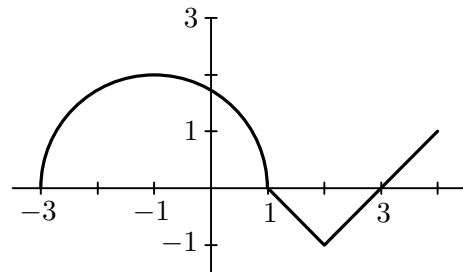


- Assuming the rocket was launched from ground level, about how high did it go?
- Assuming the rocket was launched from ground level, about how high was the rocket 12 seconds after launch?
- What is the rocket's acceleration at  $t = 6$  seconds? At  $t = 2$  seconds?

**1091.** The graph of a function  $f$  consists of a semicircle and two line segments as shown below.

$$\text{Let } g(x) = \int_1^x f(t) dt.$$

- Find  $g(1)$ .
- Find  $g(3)$ .
- Find  $g(-1)$ .
- Find all the values of  $x$  on the open interval  $(-3, 4)$  at which  $g$  has a relative maximum.



- Write an equation for the line tangent to the graph of  $g$  at  $x = -1$ .
- Find the  $x$ -coordinate of each point of inflection of the graph of  $g$  on the open interval  $(-3, 4)$ .
- Find the range of  $g$ .

**1092.** An automobile accelerates from rest at  $1 + 3\sqrt{t}$  miles per hour per second for 9 seconds.

- What is its velocity after 9 seconds?
- How far does it travel in those 9 seconds?

**1093.** Find the function  $f$  with derivative  $f'(x) = \sin x + \cos x$  whose graph passes through the point  $(\pi, 3)$ .

**1094 (1989BC).** Let  $f$  be a function such that  $f''(x) = 6x + 8$ .

- Find  $f(x)$  if the graph of  $f$  is tangent to the line  $3x - y = 2$  at the point  $(0, -2)$ .
- Find the average value of  $f(x)$  on the closed interval  $[-1, 1]$ .

**1095 (1999AB, Calculator).** A particle moves along the  $y$ -axis with velocity given by  $v(t) = t \sin(t^2)$  for  $t \geq 0$ .

- In which direction (up or down) is the particle moving at time  $t = 1.5$ ? Why?
- Find the acceleration of the particle at time  $t = 1.5$ . Is the velocity of the particle increasing at  $t = 1.5$ ?
- Given that  $y(t)$  is the position of the particle at time  $t$  and that  $y(0) = 3$ , find  $y(2)$ .
- Find the total distance traveled by the particle from  $t = 0$  and  $t = 2$ .

**1096 (1990BC).** Let  $f$  and  $g$  be continuous functions with the following properties:

- $g(x) = A - f(x)$  where  $A$  is a constant
- $\int_1^2 f(x) dx = \int_2^3 g(x) dx$
- $\int_2^3 f(x) dx = -3A$

- Find  $\int_1^3 f(x) dx$  in terms of  $A$ .
- Find the average value of  $g(x)$  in terms of  $A$  over the interval  $[1, 3]$ .
- Find the value of  $k$  if  $\int_0^1 f(x+1) dx = kA$ .

**1097 (1994AB, Calculator).** Let  $F(x) = \int_0^x \sin(t^2) dt$  for  $0 \leq x \leq 3$ .

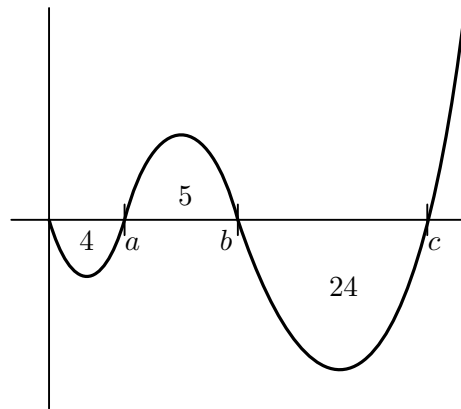
- Use the trapezoidal rule with four equal subdivisions of the closed interval  $[0, 1]$  to approximate  $F(1)$ .
- On what interval is  $F$  increasing?
- If the average rate of change of  $F$  on the closed interval  $[1, 3]$  is  $k$ , find  $\int_1^3 \sin(t^2) dt$  in terms of  $k$ .

**1098 (1991BC).** A particle moves on the  $x$ -axis so that its velocity at any time  $t \geq 0$  is given by  $v(t) = 12t^2 - 36t + 15$ .

- Find the position  $x(t)$  of the particle at any time  $t \geq 0$ .
- Find all values of  $t$  for which the particle is at rest.
- Find the maximum velocity of the particle for  $0 \leq t \leq 2$ .
- Find the total distance traveled by the particle from  $t = 0$  to  $t = 2$ .

**1099.** A particle moves along the  $x$ -axis. Its initial position at  $t = 0$  sec is  $x(0) = 15$ . The graph below shows the particle's velocity  $v(t)$ . The numbers are areas of the enclosed figures.

- What is the particle's displacement between  $t = 0$  and  $t = c$ ?
- What is the total distance traveled by the particle in the same time period?
- Give the positions of the particle at times  $a$ ,  $b$ , and  $c$ .
- Approximately where does the particle achieve its greatest positive acceleration on the interval  $[0, b]$ ? On  $[0, c]$ ?



**1100 (1987BC).** Let  $f$  be a continuous function with domain  $x > 0$  and let  $F$  be the function given by  $F(x) = \int_1^x f(t) dt$  for  $x > 0$ . Suppose that  $F(ab) = F(a) + F(b)$  for all  $a > 0$  and  $b > 0$  and that  $F'(1) = 3$ .

- Find  $f(1)$ .
- Prove that  $aF'(ax) = F'(x)$  for every positive constant  $a$ .
- Use the results from parts (a) and (b) to find  $f(x)$ . Justify your answer.

**1101 (1999AB, Calculator).** The rate at which water flows out of a pipe, in gallons per hour, is given by a differentiable function  $R$  of time  $t$ . The table below shows the rate as measured every 3 hours for a 24-hour period.

$t$ (hours)	0	3	6	9	12	15	18	21	24
$R(t)$ (gal/hr)	9.6	10.4	10.8	11.2	11.4	11.3	10.7	10.2	9.6

- Use a midpoint Riemann sum with 4 subdivisions of equal length to approximate the value of  $\int_0^{24} R(t) dt$ . Using correct units, explain the meaning of your answer in terms of water flow.
- Is there some time  $t$ ,  $0 < t < 24$ , such that  $R'(t) = 0$ ? Justify your answer.
- The rate of the water flow  $R(t)$  can be approximated by  $Q(t) = \frac{1}{79}(768 + 23t - t^2)$ . Use  $Q(t)$  to approximate the average rate of water flow during the 24-hour time period. Indicate units of measure.

---

E. H. Moore was presenting a paper on a highly technical topic to a large gathering of faculty and graduate students from all parts of the country. When half way through he discovered what seemed to be an error (though probably no one else in the room observed it). He stopped and re-examined the doubtful step for several minutes and then, convinced of the error, he abruptly dismissed the meeting – to the astonishment of most of the audience. It was an evidence of intellectual courage as well as honesty and doubtless won for him the supreme admiration of every person in the group – an admiration which was in no ways diminished, but rather increased, when at a later meeting he announced that after all he had been able to prove the step to be correct. —H. E. Slaught

### 4.19 Multiple Choice Problems on Integrals

**1102 (AP).** For any real number  $b$ ,  $\int_0^b |2x| dx$  is

- A)  $-b|b|$       B)  $b^2$       C)  $-b^2$       D)  $b|b|$       E) None of these

**1103 (AP).** Let  $f$  and  $g$  have continuous first and second derivatives everywhere. If  $f(x) \leq g(x)$  for all real  $x$ , which of the following must be true?

- I)  $f'(x) \leq g'(x)$  for all real  $x$   
 II)  $f''(x) \leq g''(x)$  for all real  $x$   
 III)  $\int_0^1 f(x) dx \leq \int_0^1 g(x) dx$

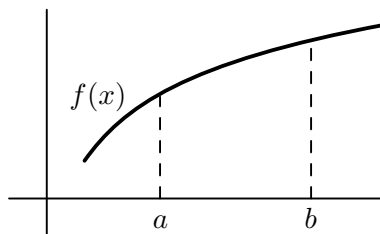
- A) None      B) I only      C) III only      D) I and II      E) I, II, and III

**1104 (AP).** Let  $f$  be a continuous function on the closed interval  $[0, 2]$ . If  $2 \leq f(x) \leq 4$ , then the greatest possible value of  $\int_0^2 f(x) dx$  is

- A) 0      B) 2      C) 4      D) 8      E) 16

**1105 (AP).** If  $f$  is the continuous, strictly increasing function on the interval  $[a, b]$  as shown below, which of the following must be true?

- I)  $\int_a^b f(x) dx < f(b)(b - a)$   
 II)  $\int_a^b f(x) dx > f(a)(b - a)$   
 III)  $\int_a^b f(x) dx = f(c)(b - a)$  for some  $c$  in  $[a, b]$ .



- A) I only      B) II only      C) III only      D) I and II      E) I, II, and III

**1106 (AP).** Which of the following definite integrals is *not* equal to zero?

- A)  $\int_{-\pi}^{\pi} \sin^3 x dx$       B)  $\int_{-\pi}^{\pi} x^2 \sin x dx$       C)  $\int_0^{\pi} \cos x dx$   
 D)  $\int_{-\pi}^{\pi} \cos^3 x dx$       E)  $\int_{-\pi}^{\pi} \cos^2 x dx$



**1107.**  $\int_{\pi/6}^{\pi/2} \cot x \, dx =$

- A)
- $\ln \frac{1}{2}$
- B)
- $\ln 2$
- C)
- $\frac{1}{2}$
- D)
- $\ln(\sqrt{3} - 1)$
- E) None of these

**1108.**  $\int_{-2}^3 |x + 1| \, dx =$

- A)
- $\frac{5}{2}$
- B)
- $\frac{17}{2}$
- C)
- $\frac{9}{2}$
- D)
- $\frac{11}{2}$
- E)
- $\frac{13}{2}$

**1109.**  $\int_1^2 (3x - 2)^3 \, dx =$

- A)
- $\frac{16}{3}$
- B)
- $\frac{63}{4}$
- C)
- $\frac{13}{3}$
- D)
- $\frac{85}{4}$
- E) None of these

**1110.**  $\int_{\pi/4}^{\pi/2} \sin^3 \theta \cos \theta \, d\theta =$

- A)
- $\frac{3}{16}$
- B)
- $\frac{1}{8}$
- C)
- $-\frac{1}{8}$
- D)
- $-\frac{3}{16}$
- E)
- $\frac{3}{4}$

**1111.**  $\int_0^1 \frac{e^x}{(3 - e^x)^2} \, dx =$

- A)
- $3 \ln(e - 3)$
- B) 1      C)
- $\frac{1}{3 - e}$
- D)
- $\frac{e - 1}{2(3 - e)}$
- E)
- $\frac{e - 2}{3 - e}$

**1112.**  $\int_{-1}^0 e^{-x} \, dx =$

- A)
- $1 - e$
- B)
- $\frac{1 - e}{e}$
- C)
- $e - 1$
- D)
- $1 - \frac{1}{e}$
- E)
- $e + 1$

**1113.**  $\int_0^1 \frac{x}{x^2 + 1} \, dx =$

- A)
- $\frac{\pi}{4}$
- B)
- $\ln \sqrt{2}$
- C)
- $\frac{1}{2}(\ln 2 - 1)$
- D)
- $\frac{3}{2}$
- E)
- $\ln 2$

**1114.** The acceleration of a particle moving along a straight line is given by  $a = 6t$ . If, when  $t = 0$  its velocity  $v = 1$  and its distance  $s = 3$ , then at any time  $t$  the position function is given by

- A)  $s = t^3 + 3t + 1$
- B)  $s = t^3 + 3$
- C)  $s = t^3 + t + 3$
- D)  $s = \frac{1}{3}t^3 + t + 3$
- E)  $s = \frac{1}{3}t^3 + \frac{1}{2}t^2 + 3$

**1115.** If the displacement of a particle on a line is given by  $s = 3 + (t - 2)^4$ , then the number of times the particle changes direction is

- A) 0
- B) 1
- C) 2
- D) 3
- E) None of these

**1116.**  $\int_0^{\pi/2} \cos^2 x \sin x \, dx =$

- A)  $-1$
- B)  $-\frac{1}{3}$
- C)  $0$
- D)  $\frac{1}{3}$
- E)  $1$

**1117.**  $\int_0^1 (3x^2 - 2x + 3) \, dx =$

- A) 0
- B) 5
- C) 3
- D) 8
- E) None of these

**1118.**  $\int_1^e \left( x - \frac{1}{2x} \right) \, dx =$

- A)  $\frac{1}{2}e^2$
- B)  $\frac{1}{2}e^2 + 1$
- C)  $\frac{1}{2}(e^2 + 1)$
- D)  $\frac{1}{2}(e^2 - 1)$
- E) None of these

**1119.**  $\int_0^1 (2 - 3x)^5 \, dx =$

- A)  $-\frac{1}{2}$
- B)  $\frac{1}{6}$
- C)  $\frac{1}{2}$
- D)  $-\frac{1}{18}$
- E) None of these

**A.P. Calculus Test Four**  
**Section One**  
**Multiple-Choice**  
**No Calculators**  
Time—35 minutes  
Number of Questions—15

The scoring for this section is determined by the formula

$$[C - (0.25 \times I)] \times 1.8$$

where  $C$  is the number of correct responses and  $I$  is the number of incorrect responses. An unanswered question earns zero points. The maximum possible points earned on this section is 27, which represents 50% of the total test score.

*Directions:* Solve each of the following problems, using the available space for scratch work. After examining the form of the choices, decide which is the best of the choices given and fill in the corresponding choice on your answer sheet. Do not spend too much time on any one problem.

Good Luck!

NAME:

1.  $\int \sin 3\theta \, d\theta =$

- A)  $3 \cos 3\theta + C$
  - B)  $-3 \cos 3\theta + C$
  - C)  $-\cos 3\theta + C$
  - D)  $\frac{1}{3} \cos 3\theta + C$
  - E)  $-\frac{1}{3} \cos 3\theta + C$
- 

2.  $\int 3^{x^2} x \, dx =$

- A)  $\frac{3^{x^2+1}}{x^2+1} + C$
  - B)  $\frac{3^{x^2}}{\ln 9} + C$
  - C)  $3^{x^2} \ln 3 + C$
  - D)  $3^{x^3/3} + C$
  - E) None of these
- 

3. Let  $f(x)$  be defined as below. Evaluate  $\int_0^6 f(x) \, dx$ .

$$f(x) = \begin{cases} x & 0 < x \leq 2 \\ 1 & 2 < x \leq 4 \\ \frac{1}{2}x & 4 < x \leq 6 \end{cases}$$

- A) 5
- B) 6
- C) 7
- D) 8
- E) 9

4.  $\int_0^1 \frac{x}{x^2 + 1} dx =$

- A)  $\frac{\pi}{4}$
  - B)  $\ln \sqrt{2}$
  - C)  $\frac{1}{2}(\ln 2 - 1)$
  - D)  $\frac{3}{2}$
  - E)  $\ln 2$
- 

5. The average value of  $g(x) = (x - 3)^2$  in the interval  $[1, 3]$  is

- A) 2
  - B)  $\frac{2}{3}$
  - C)  $\frac{4}{3}$
  - D)  $\frac{8}{3}$
  - E) None of these
- 

6.  $\int_0^5 \frac{dx}{\sqrt{3x + 1}} =$

- A)  $\frac{1}{2}$
- B)  $\frac{2}{3}$
- C) 1
- D) 2
- E) 6

7. There is a point between  $P(1, 0)$  and  $Q(e, 1)$  on the graph of  $y = \ln x$  such that the tangent to the graph at that point is parallel to the line through points  $P$  and  $Q$ . The  $x$ -coordinate of this point is

- A)  $e - 1$
  - B)  $e$
  - C)  $-1$
  - D)  $\frac{1}{e - 1}$
  - E)  $\frac{1}{e + 1}$
- 

8. Which of the following statements are true?

- I. If the graph of a function is always concave up, then the left-hand Riemann sums with the same subdivisions over the same interval are always less than the right-hand Riemann sum.
- II. If the function  $f$  is continuous on the interval  $[a, b]$  and  $\int_a^b f(x) dx = 0$ , then  $f$  must have at least one zero between  $a$  and  $b$ .
- III. If  $f'(x) > 0$  for all  $x$  in an interval, then the function  $f$  is concave up in that interval.

- A) I only
  - B) II only
  - C) III only
  - D) II and III only
  - E) None are true.
- 

9. If  $f(x) = \int_2^{2x} \frac{1}{\sqrt{t^3 + 1}} dt$ , then  $f'(1) =$

- A) 0
- B)  $\frac{1}{3}$
- C)  $\frac{2}{3}$
- D)  $\sqrt{2}$
- E) undefined

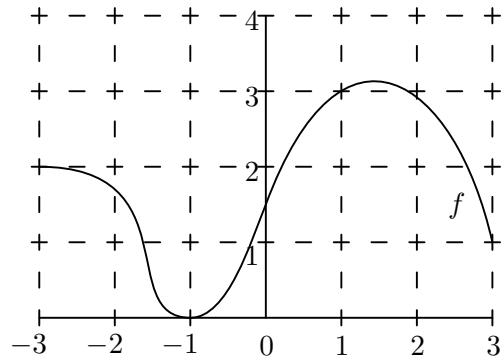
10. If  $\int_a^b f(x) dx = 3$  and  $\int_a^b g(x) dx = -2$ , then which of the following must be true?

- I.  $f(x) > g(x)$  for all  $a \leq x \leq b$
- II.  $\int_a^b [f(x) + g(x)] dx = 1$
- III.  $\int_a^b [f(x)g(x)] dx = -6$

- A) I only
- B) II only
- C) III only
- D) II and III only
- E) I, II, and III

11. The graph of  $f$  is shown below. Approximate  $\int_{-3}^3 f(x) dx$  using the trapezoid rule with 3 equal subdivisions.

- A)  $\frac{9}{4}$
- B)  $\frac{9}{2}$
- C) 9
- D) 18
- E) 36

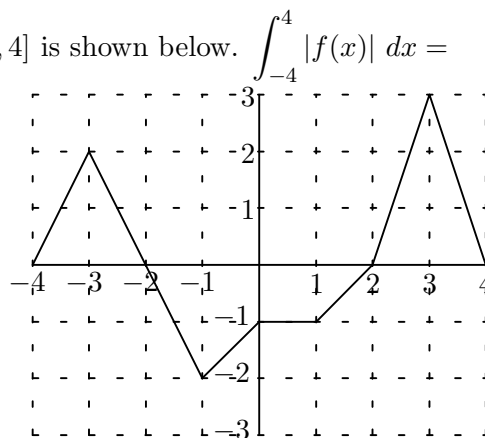


12. If  $\int_0^k \frac{\sec^2 x}{1 + \tan x} dx = \ln 2$ , then the value of  $k$  is

- A)  $\pi/6$ .
- B)  $\pi/4$ .
- C)  $\pi/3$ .
- D)  $\pi/2$ .
- E)  $\pi$ .

13. The graph of the function  $f$  on the interval  $[-4, 4]$  is shown below.  $\int_{-4}^4 |f(x)| dx =$

- A) 1  
 B) 2  
 C) 5  
 D) 8  
 E) 9



14. The acceleration of a particle moving along the  $x$ -axis at time  $t > 0$  is given by  $a(t) = \frac{1}{t^2}$ . When  $t = 1$  second, the particle is at  $x = 2$  and has velocity  $-1$  unit per second. If  $x(t)$  is the particle's position, then the position when  $t = e$  seconds is

- A)  $x = -2$ .  
 B)  $x = -1$ .  
 C)  $x = 0$ .  
 D)  $x = 1$ .  
 E)  $x = 2$ .

15. The area enclosed by the two curves  $y = x^2 - 4$  and  $y = x - 4$  is given by

- A)  $\int_0^1 (x - x^2) dx$   
 B)  $\int_0^1 (x^2 - x) dx$   
 C)  $\int_0^2 (x - x^2) dx$   
 D)  $\int_0^2 (x^2 - x) dx$   
 E)  $\int_0^4 (x^2 - x) dx$



**A.P. Calculus Test Four**  
**Section Two**  
**Free-Response**  
**Calculators Allowed**  
Time—45 minutes  
Number of Questions—3

Each of the three questions is worth 9 points. The maximum possible points earned on this section is 27, which represents 50% of the total test score. There is no penalty for guessing.

- **SHOW ALL YOUR WORK.** You will be graded on the methods you use as well as the accuracy of your answers. Correct answers without supporting work may not receive full credit.
- Write all work for each problem in the space provided. Be sure to write clearly and legibly. Erased or crossed out work will not be graded.
- Justifications require that you give mathematical (non-calculator) reasons and that you clearly identify functions, graphs, tables, or other objects that you use.
- You are permitted to use your calculator to solve an equation or graph a function without showing work. However, you must clearly indicate the setup of your problem.
- Your work must be expressed in mathematical notation rather than calculator syntax. For example,  $\int_1^5 x^2 dx$  may not be written as `fnInt(X^2,X,1,5)`.
- Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Good Luck!

NAME:

1. The temperature on New Year's Day in Buffalo, New York, is given by

$$T(h) = -A - B \cos\left(\frac{\pi h}{12}\right),$$

where  $T$  is the temperature in degrees Fahrenheit and  $h$  is the number of hours from midnight ( $0 \leq h \leq 24$ ).

- The initial temperature at midnight was  $-15^\circ$  F, and at Noon of New Year's Day it was  $5^\circ$  F. Find  $A$  and  $B$ .
  - Find the average temperature for the first 10 hours.
  - Use the trapezoid rule with 4 equal subdivisions to estimate  $\int_6^{10} T(h) dh$ . Using correct units, explain the meaning of your answer.
  - Find an expression for the rate that the temperature is changing with respect to  $h$ .
- 

2. Let  $f$  be a differentiable function, defined for all real numbers  $x$ , with the following properties.

- $f'(x) = ax^2 + bx$ , where  $a$  and  $b$  are real numbers
- $f'(1) = 6$  and  $f''(1) = 18$
- $\int_1^2 f(x) dx = 18$

Find  $f(x)$ . Show your work.

---

3. A particle moves along the  $x$ -axis so that its velocity at any time  $t \geq 0$  is given by  $v(t) = 1 - \sin(2\pi t)$ .

- Find the acceleration  $a(t)$  of the particle at any time  $t$ .
- Find all values of  $t$ ,  $0 \leq t \leq 2$ , for which the particle is at rest.
- Find the position  $x(t)$  of the particle at any time  $t$  if  $x(0) = 0$ .

## CHAPTER 5

---

# APPLICATIONS of INTEGRALS

---

## 5.1 Volumes of Solids with Defined Cross-Sections

**1120.** Find the volume of the solid whose base is bounded by the circle  $x^2 + y^2 = 9$  and the cross sections perpendicular to the  $x$ -axis are squares.

**1121.** Find the volume of the solid whose base is bounded by the ellipse  $x^2 + 4y^2 = 4$  and the cross sections perpendicular to the  $x$ -axis are squares.

**1122.** Find the volume of the solid whose base is bounded by the circle  $x^2 + y^2 = 1$  and the cross sections perpendicular to the  $x$ -axis are equilateral triangles.

**1123.** Find the volume of the solid whose base is bounded by the circle  $x^2 + y^2 = 4$  and the cross sections perpendicular to the  $x$ -axis are semicircles.

**1124.** Find the volume of the solid whose base is bounded by the circle  $x^2 + y^2 = 16$  and the cross sections perpendicular to the  $x$ -axis are isosceles right triangles having the hypotenuse in the plane of the base.

**1125.** Let  $R$  be the region bounded by  $y = e^x$ ,  $y = 2$ , and  $x = 0$ . Find the volume of the solid whose base is bounded by the region  $R$  and the cross sections perpendicular to the  $x$ -axis are semicircles.

**1126.** Let  $R$  be the region bounded by  $y = e^x$ ,  $y = 2$ , and  $x = 0$ . Find the volume of the solid whose base is bounded by the region  $R$  and the cross sections perpendicular to the  $x$ -axis are quartercircles.

**1127.** Let  $R$  be the region bounded by  $y = x^2$  and  $y = x$ . Find the volume of the solid whose base is bounded by the region  $R$  and the cross sections perpendicular to the  $x$ -axis are semicircles.

**1128.** Let  $R$  be the region bounded by  $y = \frac{1}{16}x^2$  and  $y = 2$ . Find the volume of the solid whose base is bounded by the region  $R$  and the cross sections perpendicular to the  $x$ -axis are rectangles whose height is twice that of the side in the plane of the base.

**1129.** Find the volume of the solid whose base is bounded by the curve  $y = 2\sqrt{\sin x}$ , the lines  $x = 0$ ,  $x = \pi$ , and  $y = 0$ , and the cross sections perpendicular to the  $x$ -axis are a) equilateral triangles; b) squares.

**1130.** Find the volume of the solid whose base is bounded by the curve  $y = 2x^3$ , the line  $x = 2$  and the line  $y = 0$ , and the cross sections perpendicular to the  $x$ -axis are equilateral triangles.

**1131.** Find the volume of the solid whose base is bounded by the curve  $y = 2x^3$ , the line  $x = 2$  and the line  $y = 0$ , and the cross sections perpendicular to the  $y$ -axis are equilateral triangles.

## 5.2 Turn Up the Volume!

SKETCH THE REGION  $R$  BOUNDED BY THE GIVEN CURVES AND LINES. THEN FIND THE VOLUME OF THE SOLID GENERATED BY REVOLVING  $R$  AROUND THE GIVEN AXIS.

**1132.**  $y = -2/x$ ,  $y = 1$ ,  $y = 2$ ,  $x = 0$ ; axis:  $x = 0$

**1133.**  $y = x^2$ ,  $y = 2 - x^2$ ; axis:  $y$ -axis

**1134.**  $y = \cos x$ ,  $y = \sin x$ ,  $x = 0$ ,  $x = \pi/4$ ; axis:  $x$ -axis

**1135.**  $y = x^2$ ,  $y = 0$ ,  $x = 2$ ; axis:  $y$ -axis

**1136.**  $y = 1/x^2$ ,  $x = e$ ,  $x = e^3$ ,  $y = 0$ ; axis:  $x$ -axis

**1137.**  $y = 1/x^2$ ,  $x = e$ ,  $x = e^3$ ,  $y = 0$ ; axis:  $y$ -axis

**1138.**  $y = 3 - x^2$ ,  $y = -1$ ; axis:  $y = -1$

**1139.**  $x = 1 - y^2$ ,  $x = -3$ ; axis:  $x = -3$

**1140.**  $y = 16x - 4x^2$ ,  $y = 0$ ; axis:  $y = -20$

**1141.**  $y = (x + 3)^3$ ,  $y = 0$ ,  $x = 2$ ; axis:  $y = -1$

SET UP THE INTEGRALS THAT REPRESENT THE VOLUME OF THE SOLID DESCRIBED IN THE FOLLOWING PROBLEMS. THEN USE YOUR CALCULATOR TO EVALUATE THE INTEGRALS.

**1142.** The region  $R$  is bounded by the curve  $y = -\frac{1}{2}x^3$  and the lines  $y = 4$  and  $x = 1$ . Find the volume of the solid generated by revolving  $R$  about the axis

a)  $x = 2$

b)  $y = 5$

c)  $x = -3$

d)  $y = -\frac{3}{2}$

**1143.** The region  $R$  is bounded by the curve  $y = \sin x \cos x$  and the  $x$ -axis from  $x = 0$  to  $x = \pi/2$ . Find the volume of the solid generated by revolving  $R$  about the  $x$ -axis.

**1144.** The region  $R$  is bounded by the curve  $y = \ln x$  and the lines  $y = 0$  and  $x = e^3$ . Find the volume of the solid generated by revolving  $R$  about the  $y$ -axis.

**1145.** The region  $R$  is bounded by the curve  $y = e^x$  and the lines  $y = 2$  and  $x = -1$ . Find the volume of the solid generated by revolving  $R$  about the line  $y = e$ .

**1146.** The region  $R$  is bounded by the curve  $16y^2 + 9x^2 = 144$  and the line  $4y = 3x + 12$  in Quadrant II. Find the volume of the solid generated by revolving  $R$  about the  $x$ -axis.

**1147.** The arch  $y = \sin x$ ,  $0 \leq x \leq \pi$ , is revolved about the line  $y = c$ , for  $0 \leq c \leq 1$ , to generate a solid. Find the value of  $c$  that minimizes the volume of the solid. What is the minimum volume? What value of  $c$  in  $[0, 1]$  maximizes the volume of the solid?

---

In the index to the six hundred odd pages of Arnold Toynbee's *A Study of History*, abridged version, the names of Copernicus, Galileo, Descartes and Newton do not occur yet their cosmic quest destroyed the medieval vision of an immutable social order in a walled-in universe and transformed the European landscape, society, culture, habits and general outlook, as thoroughly as if a new species had arisen on this planet. —*Arthur Koestler*

### 5.3 Volume and Arc Length

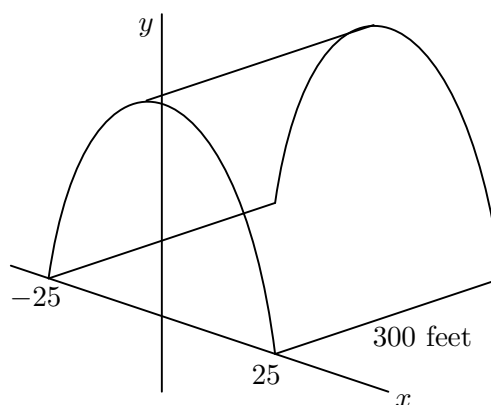
**1148.** Find the length of the curve  $x^2 + y^2 = 1$  using two different approaches. One of the techniques must involve an integral.

**1149.** Set up, *but do not evaluate*, an integral that would represent the length of the ellipse  $9x^2 + 4y^2 = 36$  in Quadrant I.

**1150.** Set up, *but do not evaluate*, an integral that would represent the length of the hyperbola  $4x^2 - 25y^2 = 100$  in Quadrant I from  $x = 0$  to  $x = 7$ .

**1151.** Set up, *but do not evaluate*, an integral that would represent the length of the curve  $y = \int_0^x \tan t \, dt$  from  $x = 0$  to  $x = \pi/6$ .

**1152.** Your engineering firm is bidding for the contract to construct the tunnel shown in the figure. The tunnel is 300 ft long and 50 ft across at the base. It is shaped like one arch of the curve  $y = 25 \cos \frac{\pi x}{50}$ . Upon completion, the tunnel's inside surface (excluding the roadway) will be coated with a waterproof sealant that costs \$1.75 per square foot to apply. How much will it cost to apply the sealant?



FIND THE EXACT LENGTH OF THE GIVEN CURVE.

**1153.**  $y = x^{3/2}$  from  $x = 0$  to  $x = 3$

**1154.**  $y = \frac{2}{3}(x + 3)^{3/2}$  from  $x = 1$  to  $x = 6$

**1155.**  $y = \frac{1}{3}(x^2 + 2)^{3/2}$  from  $x = 0$  to  $x = 3$

**1156.**  $y = \frac{3}{4}x^{4/3} - \frac{3}{8}x^{2/3} + 5$  from  $x = 1$  to  $x = 8$

**1157.**  $y = \int_{-2}^x \sqrt{3t^4 - 1} \, dt$  from  $x = -2$  to  $x = 1$

SKETCH THE REGION  $R$  BOUNDED BY THE GIVEN CURVES, LINES, AND THE  $x$ -AXIS. THEN FIND THE VOLUME OF THE SOLID GENERATED BY REVOLVING  $R$  AROUND THE  $x$ -AXIS.

**1158.**  $f(x) = \sqrt{x-2}$ ,  $x = 3$ ,  $x = 4$

**1162.**  $f(x) = 3/x$ ,  $x = e$ ,  $x = 3$

**1159.**  $f(x) = x^3 + 8$ ,  $x = 0$

**1163.**  $f(x) = 2 \cos 3x$ ,  $x = \pi/6$ ,  $x = \pi/3$

**1160.**  $f(x) = \sin 2x$ ,  $x = 0$ ,  $x = \pi$

**1164.**  $f(x) = x^2$ ,  $g(x) = x$

**1161.**  $f(x) = \frac{1}{2}e^{x/8}$ ,  $x = \ln 16$ ,  $x = \ln 81$

**1165.**  $f(x) = 1/\sqrt{x}$ ,  $x = 1$ ,  $x = e$

## 5.4 Differential Equations, Part One

FIND THE FUNCTION WITH THE GIVEN DERIVATIVE WHOSE GRAPH PASSES THROUGH THE POINT  $P$ .

**1166.**  $f'(x) = 2x - 1$ ,  $P(0, 0)$

**1169.**  $r'(t) = \sec t \tan t - 1$ ,  $P(0, 0)$

**1167.**  $g'(x) = \frac{1}{x} + 2x$ ,  $P(1, -1)$

**1170.**  $s'(t) = 9.8t + 5$ ,  $P(0, 10)$

**1168.**  $f'(x) = e^{2x}$ ,  $P(0, \frac{3}{2})$

**1171.**  $s'(t) = 32t - 2$ ,  $P(0.5, 4)$

GIVEN THE ACCELERATION, INITIAL VELOCITY, AND INITIAL POSITION OF A PARTICLE, FIND THE PARTICLE'S POSITION AT ANY TIME  $t$ .

**1172.**  $a(t) = e^t$ ,  $v(0) = 20$ ,  $s(0) = 5$

**1173.**  $a(t) = -4 \sin(2t)$ ,  $v(0) = 2$ ,  $s(0) = -3$

FIND THE GENERAL SOLUTION TO THE GIVEN DIFFERENTIAL EQUATION.

**1174.**  $\frac{dy}{dx} = 2x + 7$

**1177.**  $\frac{du}{dv} = 2u^4v$

**1175.**  $\frac{dy}{dx} = 4x^3 + 2x - 1$

**1178.**  $\frac{dy}{dx} = 2\sqrt{x}$

**1176.**  $\frac{dr}{dt} = 4t^3r$

**1179.**  $\frac{dy}{dx} = 2(3x + 5)^3$

FIND THE PARTICULAR SOLUTION TO THE GIVEN DIFFERENTIAL EQUATION.

**1180.**  $\frac{ds}{dt} = \cos t + \sin t$ ,  $s(\pi) = 1$

**1183.**  $\frac{dv}{dt} = \frac{8}{1+t^2} + \sec^2 t$ ,  $v(0) = 1$

**1181.**  $\frac{dr}{d\theta} = -\pi \sin(\pi\theta)$ ,  $r(0) = 0$

**1184.**  $\frac{d^2y}{dx^2} = 2 - 6x$ ,  $y'(0) = 4$ ,  $y(0) = 1$

**1182.**  $\frac{dv}{dt} = \frac{3}{t\sqrt{t^2-1}}$ ,  $v(2) = 0$ ,  $t > 1$

**1185.**  $\frac{d^2y}{dx^2} = \frac{2}{x^3}$ ,  $y'(1) = 1$ ,  $y(1) = 1$

**1186.** *A Curious Property of Definite Integrals*

- a) Let  $R_1$  be the region bounded by  $f(x) = \frac{1}{x}$ ,  $x = 1$ ,  $x = 3$ , and the  $x$ -axis. Draw the region  $R_1$  and find the area of  $R_1$  using an integral.
- b) Let  $R_2$  be the region bounded by  $f(x) = \frac{1}{x-2}$ ,  $x = 3$ ,  $x = 5$ , and the  $x$ -axis. Draw the region  $R_2$  and find the area of  $R_2$  using an integral.
- c) What do you notice about your answers in parts (a) and (b)?
- d) Complete the following conjecture, where  $a$ ,  $b$ , and  $c$  are real numbers: *If  $f(x)$  is a continuous function on  $[a, b]$ , then  $\int_a^b f(x) dx = \int_{a+c}^{b+c}$  \_\_\_\_\_  $dx$ .*

## 5.5 The Logistic Curve

**1187.** The graph of a function of the form  $P(t) = \frac{M}{1 + Ce^{-rMt}}$ , where  $M$ ,  $r$ , and  $C$  are constants, is called a *logistic curve*. Graph the function  $y(x) = \frac{8}{1 + 10e^{-0.9x}}$  in the window  $-1 \leq x \leq 10$ ,  $-1 \leq y \leq 9$ . What value does  $y$  approach as  $x \rightarrow \infty$ ? What appears to be the  $y$ -value of the point where  $dy/dt$  is changing the fastest?

**1188.** The solution to the differential equation  $\frac{dP}{dt} = r(M - P)P$  is a logistic curve, where  $C$  is determined by the initial condition. Can the values found in the previous problem be found without solving the differential equation? In other words, in the equation  $dP/dt = 0.001(100 - P)P$ , what does  $P$  approach as  $x \rightarrow \infty$ ? What appears to be the  $P$ -value of the point where  $dP/dt$  is changing the fastest?

**1189.** A 2000 gallon tank can support no more than 150 guppies. Six guppies are introduced into the tank. Assume that the rate of growth of the population is  $dP/dt = 0.0015(150 - P)P$ , where  $t$  is in weeks. Find a formula for the guppy population in terms of  $t$ ; then, determine how long it will take for the guppy population to be 100.

**1190.** A certain wild animal preserve can support no more than 250 gorillas. In 1970, 28 gorillas were known to be in the preserve. Assume that the rate of growth of population is  $dP/dt = 0.0004(250 - P)P$ , where  $t$  is in years. Find a formula for the gorilla population in terms of  $t$ ; then, determine how long it will take for the gorilla population to reach the carrying capacity of the preserve. What is the gorilla population when the rate of change of the population is maximized?

**1191.** Solve the differential equation  $dP/dt = kP^2$  for constant  $k$ , with initial condition  $P(0) = P_0$ . Prove that the graph of the solution has a vertical asymptote at a positive value of  $t$ . What is that value of  $t$ ? (This value is called the *catastrophic solution*.)

**1192.** Given a differential equation of the form  $ay'' + by' + y = 0$ , find constants  $a$  and  $b$  so that both  $y = e^x$  and  $y = e^{2x}$  are solutions.

**1193 (AP).** At each point  $(x, y)$  on a certain curve, the slope of the curve is  $3x^2y$ . If the curve contains the point  $(0, 8)$ , then its equation is

- A)  $y = 8e^{x^3}$
- B)  $y = x^3 + 8$
- C)  $y = e^{x^3} + 7$
- D)  $y = \ln(x + 1) + 8$
- E)  $y^2 = x^2 + 8$

---

The simplest schoolboy is now familiar with facts for which Archimedes would have sacrificed his life.  
—*Earnest Renan*



## 5.6 Differential Equations, Part Two

**1194.** You are driving along the highway at a steady 60 mph (88 ft/sec) when you see an accident ahead and slam on the brakes. What constant deceleration is required to stop your car in 242 ft?

**1195.** The rate of change in the number of bacteria in a culture is proportional to the number present. AP Biology students at Rockdale discovered that there were 3000 bacteria initially, and 90,000 bacteria after two hours.

- In terms of  $t$  only, find the number of bacteria in the culture at any time  $t$ .
- How many bacteria were there after four hours?
- How many hours have elapsed when the students observed 60,000 bacteria in the culture?

**1196.** The rate of increase of the population of Springfield is proportional to the population at any given time. If the population in 1950 was 50,000 and in 1980 it was 75,000, what is the expected population in the year 2010? When will Springfield's population reach 1,000,000 people? Justify your answer.

**1197.** Corbin's hobby is to buy antique cars, repair them, and then sell them at a good profit. Research shows that the rate of change in the value of Corbin's cars is directly proportional to the value of the car at any given time. If Corbin bought a 1945 Jaguar from his aunt for \$49,000 in 2002, and if the Jaguar's market value in 2008 is \$63,000, what is the expected value of the Jaguar in the year 2014? How long will Corbin have to wait for the Jaguar's market value to be \$100,000?

**1198.** Oil is being pumped continuously from an Arabian oil well at a rate proportional to the amount of oil left in the well; that is,  $dy/dt = ky$  where  $y$  is the number of gallons of oil left in the well at any time  $t$  (in years). Initially there are 1,000,000 gallons of oil in the well, and 6 years later there are 500,000 remaining. Assume that it is no longer profitable to pump oil when there are fewer than 50,000 gallons remaining.

- Write an equation for  $y$  in terms of  $t$ .
- At what rate is the amount of oil in the well decreasing when there are 600,000 gallons of oil remaining in the well?
- How long will it be profitable to pump oil?

**1199.** When stated in the form of a differential equation, *Newton's Law of Cooling* becomes  $dT/dt = -k(T - T_a)$ , where  $k$  is a positive constant and  $T_a$  is the ambient temperature.

- Find the general solution for  $T$ , satisfying the initial condition  $T(0) = T_0$ .
- What is the limiting temperature as  $t$ ? Explain the difference between what happens when  $T_0 \leq T_a$ , and when  $T_0 \geq T_a$ .
- A 15-pound roast, initially at 60°F, is put into a 350°F oven at 2 PM. After two hours, the temperature of the roast is 120°F. When will the roast be at a temperature of 150°F?

## 5.7 Slope Fields and Euler's Method

- 1200.** Consider the differential equation  $\frac{dy}{dx} = \frac{2x}{y}$ .
- Sketch a slope field for the equation at the points  $(-1, 1)$ ,  $(-1, 2)$ ,  $(-1, 3)$ ,  $(0, 1)$ ,  $(0, 2)$ ,  $(0, 3)$ ,  $(1, 1)$ ,  $(1, 2)$ , and  $(1, 3)$ .
  - Use the slope field program on your calculator to generate a slope field.
  - Solve the equation and sketch the solution curve through the points  $(4, 6)$  and  $(-4, 6)$ .
- 1201.** Consider the ellipse  $4x^2 + 9y^2 = 36$ .
- Find  $dy/dx$ .
  - Graph the slope field for the differential equation found in part (a) using your calculator.
  - Graph the particular solution passing through  $(3, 0)$ .
- 1202.** Let  $\frac{dy}{dx} = -\frac{2x}{y}$  be a differential equation that contains the point  $(0, 2\sqrt{2})$ .
- Approximate 6 points in the particular solution to the above equation using Euler's Method. Use 0.2 as the step size. Do not use your calculator program.
  - Repeat part (a) with a step size of 0.1, but this time, use your calculator program.
  - Solve the equation analytically. Compare the actual  $y$ -values with those obtained using Euler's Method. What conclusion could you draw?
- 1203.** Let  $\frac{dy}{dx} = \frac{3}{x}$  be subject to the initial condition that  $y(1) = 0$ .
- Use Euler's Method with step size 0.25 to approximate  $y(2)$ . Do not use your calculator.
  - Solve the equation and calculate the exact value of  $y(2)$ .
  - Graph the slope field for the equation and use it to determine if your answer in part (a) is greater than or less than the value obtained in part (b).
- 1204.** The *normal probability density function* is very important in statistics and is defined by  $G(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2/2} dt$ .
- Find  $G'(x)$ .
  - Use the slope field program to help you sketch the slope field for  $G'(x)$ . Use a window of  $-4 \leq x \leq 4$  by  $-1 \leq y \leq 1$ .
  - Sketch the solution curve through the point  $(0, \frac{1}{2})$ .

## 5.8 Differential Equations, Part Three

**1205.** The growth rate of an evergreen shrub during its first 6 years is approximated by  $dh/dt = 1.5t + 6$ , where  $t$  is the time in years and  $h$  is the height in centimeters. The seedlings are 12 cm tall when planted ( $t = 0$ ). Find the height after  $t$  years; then, determine the height of the shrubs after 6 years.

**1206.** The rate of growth of a population of bacteria is proportional to the square root of  $t$ , where  $P$  is the population and  $t$  is the time in days for  $0 \leq t \leq 10$ . The initial size of the population is 500. After 1 day, the population has grown to 600. Estimate the population after 7 days.

**1207.** Suppose that rabbits are introduced onto an island where they have no natural enemies. Because of natural conditions, the island can support a maximum of 1000 rabbits. Let  $P(t)$  denote the number of rabbits at time  $t$  (measured in months), and suppose that the population varies in size (due to births and deaths) at a rate proportional to both  $P(t)$  and  $1000 - P(t)$ . That is, suppose that  $P(t)$  satisfies the differential equation  $dP/dt = kP(1000 - P)$ , where  $k$  is a positive constant.

- Find the value of  $P$  when the rate of change of the rabbit population is maximized.
- When is the rate of change of the rabbit population a minimum? Discuss your answers.
- Assuming 50 rabbits were placed on the island, sketch the graph that would show how  $t$  and  $P(t)$  are related.

**1208.** Show that  $y = x^3 + x + 2 + \int_0^x \sin(t^2) dt$  is a solution to the differential equation  $y'' = 6x + 2x \cos(x^2)$  with initial conditions  $y'(0) = 1$  and  $y(0) = 2$ .

**1209.** Under some conditions, the result of the movement of a dissolved substance across a cell's membrane is described by the differential equation  $\frac{dy}{dt} = k\frac{A}{V}(c - y)$ , where  $y$  is the concentration of the substance in the cell,  $A$  is the surface area of the membrane,  $V$  is the cell's volume,  $c$  the concentration of the substance outside the cell, and  $k$  is a constant. Solve the equation with initial condition  $y(0) = y_0$ ; then, determine  $\lim_{t \rightarrow \infty} y(t)$ . (This is called the *steady state* concentration.)

## 5.9 Sample A.P. Problems on Applications of Integrals

FOR THE FOLLOWING FOUR PROBLEMS, USE EULER'S METHOD WITH THE GIVEN STEP SIZE  $h$  TO ESTIMATE THE THE VALUE OF THE SOLUTION AT THE GIVEN POINT  $x^*$ .

1210.  $y' = 2xe^{x^2}$ ,  $y(0) = 2$ ,  $h = 0.1$ ,  $x^* = 1$

1211.  $y' = y + e^x - 2$ ,  $y(0) = 2$ ,  $h = 0.5$ ,  $x^* = 2$

1212.  $y' = y^2/\sqrt{x}$ ,  $y(1) = -1$ ,  $h = 0.5$ ,  $x^* = 5$

1213.  $y' = y - e^{2x}$ ,  $y(0) = 1$ ,  $h = \frac{1}{3}$ ,  $x^* = 2$

1214. Let  $R$  represent the area in Quadrant IV bounded by  $f(x) = x^3 - 4x$  and  $g(x) = 0$ .

- Find the area of  $R$ .
- Find the volume of the solid generated by revolving  $R$  around the  $x$ -axis.
- Find the average value of  $f(x)$  over the interval  $[-3, -2]$ .

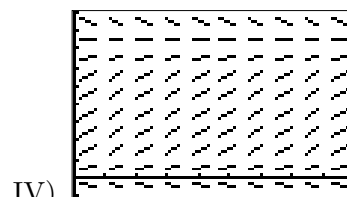
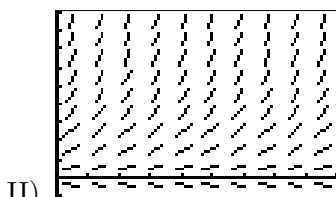
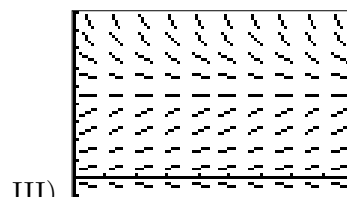
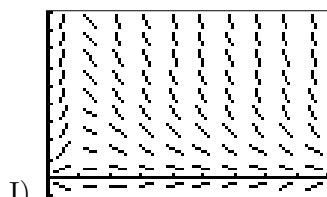
1215. Match the differential equation with its slope field.

a)  $\frac{dy}{dx} = 0.065y$

b)  $\frac{dy}{dx} = 0.06y \left(1 - \frac{y}{100}\right)$

c)  $\frac{dy}{dx} = \frac{y}{x} - y$

d)  $\frac{dy}{dx} = 0.06y \left(1 - \frac{y}{150}\right)$



1216 (1996AB). Let  $R$  be the region in the first quadrant under the graph of  $y = \frac{1}{\sqrt{x}}$  for  $4 \leq x \leq 9$ .

- Find the area of  $R$ .
- If the line  $x = k$  divides the region  $R$  into two regions of equal area, what is the value of  $k$ ?
- Find the volume of the solid whose base is the region  $R$  and whose cross sections cut by planes perpendicular to the  $x$ -axis are squares.

1217. Use your calculator to find the length of the curve  $f(x) = \frac{x-1}{4x^2+1}$  on the interval  $[-\frac{1}{2}, 1]$ .

**1218 (1995AB, Calculator).** The region  $R_1$  is bounded above by  $g(x) = 2^x$  and below by  $f(x) = x^2$ , while the region  $R_2$  is bounded above by  $f(x) = x^2$  and bounded below by  $g(x) = 2^x$ .

- Find the  $x$ - and  $y$ -coordinates of the three points of intersection of the graphs of  $f$  and  $g$ .
- Without using absolute value, set up an expression involving one or more integrals that gives the total area enclosed by the graphs of  $f$  and  $g$ . Do not evaluate.
- Without using absolute value, set up an expression involving one or more integrals that gives the volume of the solid generated by revolving the region  $R_1$  about the line  $y = 5$ . Do not evaluate.

**1219 (1999BC, Calculator).** Let  $f$  be the function whose graph goes through the point  $(3, 6)$  and whose derivative is given by  $f'(x) = \frac{1 + e^x}{x^2}$ .

- Write an equation of the line tangent to the graph of  $f$  at  $x = 3$  and use it to approximate  $f(3.1)$ .
- Use Euler's Method, starting at  $x = 3$  with a step size of 0.05, to approximate  $f(3.1)$ . Use  $f''$  to explain why this approximation is less than  $f(3.1)$ .
- Use  $\int_3^{3.1} f'(x) dx$  to evaluate  $f(3.1)$ .

**1220.** Find the particular solution to the differential equation  $\frac{dy}{dx} = \frac{4\sqrt{y} \ln x}{x}$  with initial value  $y(e) = 1$ .

**1221 (1997BC, Calculator).** Let  $R$  be the region enclosed by the graphs of  $y = \ln(x^2 + 1)$  and  $y = \cos x$ .

- Find the area of  $R$ .
- Write an expression involving one or more integrals that gives the length of the boundary of the region  $R$ . Do not evaluate.
- The base of a solid is the region  $R$ . Each cross section of the solid perpendicular to the  $x$ -axis is an equilateral triangle. Write an expression involving one or more integrals that gives the volume of the solid. Do not evaluate.

**1222 (1993AB).** Let  $P(t)$  represent the number of wolves in a population at time  $t$  years, when  $t \geq 0$ . The population  $P$  is increasing at a rate directly proportional to  $800 - P$ , where the constant of proportionality is  $k$ .

- If  $P(0) = 500$ , find  $P(t)$  in terms of  $t$  and  $k$ .
- If  $P(2) = 700$ , find  $k$ .
- Find  $\lim_{t \rightarrow \infty} P(t)$ .

---

From the intrinsic evidence of His creation, the Great Architect of the universe now begins to appear as a pure mathematician. —*Sir James Hopwood Jeans*

**1223 (1993AB).** Consider the curve  $y^2 = 4 + x$  and chord  $\overline{AB}$  joining points  $A(-4, 0)$  and  $B(0, 2)$  on the curve.

- Find the  $x$ - and  $y$ -coordinates of the point on the curve where the tangent line is parallel to chord  $\overline{AB}$ .
- Find the area of the region  $R$  enclosed by the curve and chord  $\overline{AB}$ .
- Find the volume of the solid generated when the region  $R$  defined in part (b) is revolved about the  $x$ -axis.

**1224 (1988AB).** Let  $R$  be the region in the first quadrant enclosed by the hyperbola  $x^2 - y^2 = 9$ , the  $x$ -axis, and the line  $x = 5$ .

- Find the volume of the solid generated by revolving region  $R$  about the  $x$ -axis.
- Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid generated by revolving region  $R$  about the line  $x = -1$ .

**1225 (1991BC, Calculator).** Let  $F(x) = \int_1^{2x} \sqrt{t^2 + t} dt$ .

- Find  $F'(x)$ .
- Find the domain of  $F$ .
- Find  $\lim_{x \rightarrow 1/2} F(x)$ .
- Find the length of the curve  $y = F(x)$  for  $1 \leq x \leq 2$ .

**1226 (1989AB).** Let  $R$  be the region in the first quadrant enclosed by the graph of  $y = \sqrt{6x + 4}$ , the line  $y = 2x$ , and the  $y$ -axis.

- Find the area of  $R$ .
- Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid generated by revolving region  $R$  about the  $x$ -axis.
- Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid generated by revolving region  $R$  about the  $y$ -axis.

**1227 (1990BC).** Let  $R$  be the region in the  $xy$ -plane between the graphs of  $y = e^x$  and  $y = e^{-x}$  from  $x = 0$  to  $x = 2$ .

- Find the volume of the solid generated when  $R$  is revolved about the  $x$ -axis.
- Find the volume of the solid generated when  $R$  is revolved about the  $y$ -axis.

### 5.10 Multiple Choice Problems on Application of Integrals

1228.  $\int_1^e \frac{2}{1+3x} dx =$

- A)  $\frac{2}{3} \ln\left(\frac{1+3e}{4}\right)$   
 B)  $\frac{-1}{3(1+3e)^2} + 48$   
 C)  $2 \ln\left(\frac{1+3e}{4}\right)$   
 D)  $\frac{3}{(1+3e)^2} - \frac{3}{16}$   
 E) None of these

1230.  $\int_{\pi/4}^{\pi} \sin(2\theta) d\theta =$

- A)  $-2$                       B)  $\frac{1}{2}$                       C)  $-\frac{1}{2}$                       D)  $2$                       E)  $\frac{3}{2}\sqrt{2}$

1231. The average value of  $\sqrt{x}$  over the interval  $[0, 2]$  is

- A)  $\frac{1}{3}\sqrt{2}$                       B)  $\frac{1}{2}\sqrt{2}$                       C)  $\frac{2}{3}\sqrt{2}$                       D)  $1$                       E)  $\frac{4}{3}\sqrt{2}$

1232. Estimate the area bounded by  $y = x^2$ , the  $x$ -axis, the line  $x = 1$  and the line  $x = 2$  by using a left-hand Riemann sum with 3 subintervals of equal length.

- A)  $\frac{50}{27}$                       B)  $\frac{251}{108}$                       C)  $\frac{7}{3}$                       D)  $\frac{127}{54}$                       E)  $\frac{77}{27}$

1233. If  $dy/dx = \cos(2x)$ , then  $y =$

- A)  $-\frac{1}{2} \cos(2x) + C$   
 B)  $-\frac{1}{2} \cos^2 x + C$   
 C)  $\frac{1}{2} \sin(2x) + C$   
 D)  $\frac{1}{2} \sin^2(2x) + C$   
 E)  $-\frac{1}{2} \sin(2x) + C$

1229 (AP). Which of the following integrals gives the length of the graph of  $y = \sqrt{x}$  on the interval  $[a, b]$ ?

- A)  $\int_a^b \sqrt{x^2 + x} dx$   
 B)  $\int_a^b \sqrt{x + \sqrt{x}} dx$   
 C)  $\int_a^b \sqrt{x + \frac{1}{2\sqrt{x}}} dx$   
 D)  $\int_a^b \sqrt{1 + \frac{1}{2\sqrt{x}}} dx$   
 E)  $\int_a^b \sqrt{1 + \frac{1}{4x}} dx$

**1234.** A solid is generated when the region in the first quadrant bounded by the graph of  $y = 1 + \sin^2 x$ , the line  $x = \pi/2$ , the  $x$ -axis, and the  $y$ -axis is revolved about the  $x$ -axis. Its volume is found by evaluating which of the following integrals?

A)  $\pi \int_0^1 (1 + \sin^4 x) dx$

B)  $\pi \int_0^1 (1 + \sin^2 x)^2 dx$

C)  $\pi \int_0^{\pi/2} (1 + \sin^4 x) dx$

D)  $\pi \int_0^{\pi/2} (1 + \sin^2 x)^2 dx$

E)  $\pi \int_0^{\pi/2} (1 + \sin^2 x) dx$

**1235.** The volume generated by revolving about the  $x$ -axis the region above the curve  $y = x^3$ , below the line  $y = 1$ , and between  $x = 0$  and  $x = 1$  is

A)  $\frac{\pi}{42}$

B)  $0.143\pi$

C)  $\frac{\pi}{7}$

D)  $0.857\pi$

E)  $\frac{64\pi}{7}$

**1236.** Find the distance traveled (to three decimal places) from  $t = 1$  to  $t = 5$  seconds, for a particle whose velocity is given by  $v(t) = t + \ln t$ .

A) 6.000

B) 1.609

C) 16.047

D) 0.800

E) 148.413

**1237.** A region is enclosed by the graphs of the line  $y = 2$  and the parabola  $y = 6 - x^2$ . Find the volume of the solid generated when this region is revolved about the  $x$ -axis.

A) 76.8

B) 107.2

C) 167.6

D) 183.3

E) 241.3

**1238.** Find the area of the region above the  $x$ -axis and beneath one arch of the graph of  $y = \frac{1}{2} + \sin x$ .

A)  $\frac{2}{3}\pi + \sqrt{3}$

B)  $\frac{2}{3}\pi + 1$

C)  $\sqrt{3} - \frac{1}{3}\pi$

D)  $\sqrt{3} + \frac{4}{3}\pi$

E)  $\frac{7}{12}\pi + \frac{1}{2}\sqrt{3} + 1$



**1239.** The velocity of a particle moving along the  $x$ -axis is given by  $v(t) = t \sin(t^2)$ . Find the total distance traveled from  $t = 0$  to  $t = 3$ .

- A) 1.0                      B) 1.5                      C) 2.0                      D) 2.5                      E) 3.0

**1240.** Let  $f(x)$  be a differentiable function whose domain is the closed interval  $[0, 5]$ , and let  $F(x) = \int_0^x f(t) dt$ . If  $F(5) = 10$ , which of the following must be true?

- I.  $F(x) = 2$  for some value of  $x$  in  $[0, 5]$ .  
II.  $f(x) = 2$  for some value of  $x$  in  $[0, 5]$ .  
III.  $f'(x) = 2$  for some value of  $x$  in  $[0, 5]$ .

- A) I only                      B) II only                      C) III only                      D) I and II                      E) I, II, and III

**1241.** The base of a solid is the region in the  $xy$ -plane beneath the curve  $y = \sin(kx)$  and above the  $x$ -axis for  $0 \leq x \leq \frac{\pi}{2k}$ . Each of the solid's cross-sections perpendicular to the  $x$ -axis has shape of a rectangle with height  $\cos^2(kx)$ . If the volume of the solid is 1 cubic unit, find the value of  $k$ . (Assume  $k > 0$ .)

- A) 3                      B)  $3\pi$                       C)  $\frac{1}{3\pi}$                       D)  $\frac{\pi}{3}$                       E)  $\frac{1}{3}$

**1242.** The average value of  $g(x) = (x - 3)^2$  in the interval  $[1, 3]$  is

- A) 2                      B)  $\frac{2}{3}$                       C)  $\frac{4}{3}$                       D)  $\frac{8}{3}$                       E) None of these

---

The traditional mathematics professor of the popular legend is absentminded. He usually appears in public with a lost umbrella in each hand. He prefers to face the blackboard and to turn his back to the class. He writes  $a$ , he says  $b$ , he means  $c$ ; but it should be  $d$ . Some of his sayings are handed down from generation to generation:

“In order to solve this differential equation you look at it till a solution occurs to you.”

“This principle is so perfectly general that no particular application of it is possible.”

“Geometry is the science of correct reasoning on incorrect figures.”

“My method to overcome a difficulty is to go round it.”

“What is the difference between method and device? A method is a device which you used twice.”

—George Polya

**A.P. Calculus Test Five**  
**Section One**  
**Multiple-Choice**  
**No Calculators**  
Time—35 minutes  
Number of Questions—15

The scoring for this section is determined by the formula

$$[C - (0.25 \times I)] \times 1.8$$

where  $C$  is the number of correct responses and  $I$  is the number of incorrect responses. An unanswered question earns zero points. The maximum possible points earned on this section is 27, which represents 50% of the total test score.

*Directions:* Solve each of the following problems, using the available space for scratch work. After examining the form of the choices, decide which is the best of the choices given and fill in the corresponding choice on your answer sheet. Do not spend too much time on any one problem.

Good Luck!

NAME:

1.  $\int_{\pi/4}^{\pi/2} \sin^3 \alpha \cos \alpha \, d\alpha =$

A)  $\frac{3}{16}$

B)  $\frac{1}{8}$

C)  $-\frac{1}{8}$

D)  $-\frac{3}{16}$

E)  $\frac{3}{4}$

---

2. If the distance of a particle from the origin on a line is given by  $x(t) = 3 + (t - 2)^4$ , then the number of times the particle reverses direction is

A) 0

B) 1

C) 2

D) 3

E) None of these

---

3.  $\int \tan x \, dx =$

A)  $-\ln |\sec x| + C$

B)  $\sec^2 x + C$

C)  $\ln |\sin x| + C$

D)  $\sec x + C$

E)  $-\ln |\cos x| + C$

4. Solve the differential equation  $\frac{dy}{dx} = y$  with the initial condition that  $y(0) = 1$ . From your solution, find the value of  $y(e)$ .

- A)  $e^e$
  - B)  $e$
  - C)  $e - 1$
  - D)  $e^e - e$
  - E)  $e^2$
- 

5. The average value of  $p(x) = \frac{1}{x}$  from  $x = 1$  to  $x = e$  is

- A)  $\frac{1}{e + 1}$
  - B)  $\frac{1}{1 - e}$
  - C)  $e - 1$
  - D)  $1 - \frac{1}{e}$
  - E)  $\frac{1}{e - 1}$
- 

6. The volume of a solid generated by revolving the region enclosed by the curve  $y = 3x^2$  and the line  $y = 6x$  about the  $x$ -axis is represented by

- A)  $\pi \int_0^3 (6x - 3x^2)^2 dx$
- B)  $\pi \int_0^2 (6x - 3x^2)^2 dx$
- C)  $\pi \int_0^2 (9x^4 - 36x^2) dx$
- D)  $\pi \int_0^2 (36x^2 - 9x^4) dx$
- E)  $\pi \int_0^2 (6x - 3x^2) dx$

7. A region in the plane is bounded by  $y = \frac{1}{\sqrt{x}}$ , the  $x$ -axis, the line  $x = m$ , and the line  $x = 2m$ , where  $m > 0$ . A solid is formed by revolving the region about the  $x$ -axis. The volume of this solid

- A) is independent of  $m$ .
  - B) increases as  $m$  increases.
  - C) decreases as  $m$  decreases.
  - D) increases until  $m = \frac{1}{2}$ , then decreases.
  - E) cannot be found with the information given.
- 

8. If the graph of  $y = f(x)$  contains the point  $(0, 1)$ , and if  $\frac{dy}{dx} = \frac{x \sin(x^2)}{y}$ , then  $f(x) =$

- A)  $\sqrt{2 - \cos(x^2)}$
  - B)  $\sqrt{2} - \cos(x^2)$
  - C)  $2 - \cos(x^2)$
  - D)  $\cos(x^2)$
  - E)  $\sqrt{2 - \cos x}$
- 

9.  $\lim_{h \rightarrow 0} \left( \frac{\tan(x+h) - \tan x}{h} \right) =$

- A)  $\sec x$
- B)  $-\sec x$
- C)  $\sec^2 x$
- D)  $-\sec^2 x$
- E) does not exist

10. Given the differential equation  $\frac{dy}{dx} = x + y$  with initial condition  $y(0) = 2$ , approximate  $y(1)$  using Euler's method with a step size of 0.5.

- A) 3
  - B)  $\frac{7}{2}$
  - C)  $\frac{15}{4}$
  - D)  $\frac{19}{4}$
  - E)  $\frac{21}{4}$
- 

11. The base of a solid is a right triangle whose perpendicular sides have lengths 6 and 4. Each plane section of the solid perpendicular to the side of length 6 is a semicircle whose diameter lies in the plane of the triangle. The volume of the solid is

- A)  $2\pi$  units<sup>3</sup>.
  - B)  $4\pi$  units<sup>3</sup>.
  - C)  $8\pi$  units<sup>3</sup>.
  - D)  $16\pi$  units<sup>3</sup>.
  - E)  $24\pi$  units<sup>3</sup>.
- 

12. Which of the following expressions represents the length of the curve  $y = e^{-x^2}$  for  $x$  from 0 to 2 ?

- A)  $\int_0^2 \sqrt{1 + e^{-2x^2}} dx$
- B)  $\int_0^2 \sqrt{1 + 4x^2 e^{-2x^2}} dx$
- C)  $\int_0^2 \sqrt{1 - e^{-2x^2}} dx$
- D)  $\int_0^2 \sqrt{1 + 2x e^{-2x^2}} dx$
- E)  $\pi \int_0^2 e^{-2x^2} dx$

13. If  $f(x) = \int_2^{\sin x} \sqrt{1+t^2} dt$ , then  $f'(x) =$

- A)  $(1+x^2)^{3/2}$
  - B)  $(\cos x)\sqrt{1+\sin x}$
  - C)  $\sqrt{1+\sin^2 x}$
  - D)  $(\cos x)\sqrt{1+\sin^2 x}$
  - E)  $(\cos x)(1+\sin^2 x)^{3/2}$
- 

14. For what value of  $x$  is the line tangent to  $y = x^2$  parallel to the line tangent to  $y = \sqrt{x}$ ?

- A) 0
  - B)  $\frac{1}{4\sqrt[3]{4}}$
  - C)  $\frac{1}{2}$
  - D)  $\frac{1}{2\sqrt[3]{2}}$
  - E) 1
- 

15. An antiderivative of  $(x^2 + 1)^2$  is

- A)  $\frac{1}{3}(x^2 + 1)^3 + C$
- B)  $\frac{1}{5}x^5 + x + C$
- C)  $\frac{1}{5}x^5 + \frac{2}{3}x^3 + x + C$
- D)  $\frac{1}{6x}(x^2 + 1) + C$
- E)  $4x(x^2 + 1) + C$

**A.P. Calculus Test Five**  
**Section Two**  
**Free-Response**  
**Calculators Allowed**  
Time—45 minutes  
Number of Questions—3

Each of the three questions is worth 9 points. The maximum possible points earned on this section is 27, which represents 50% of the total test score. There is no penalty for guessing.

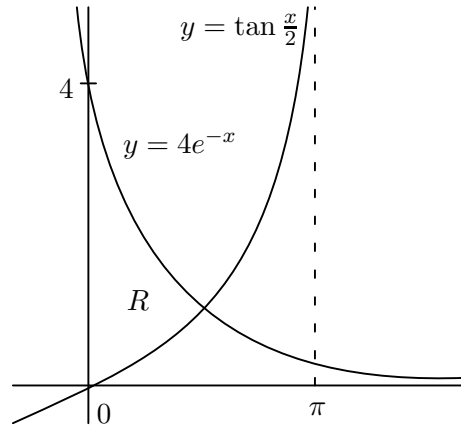
- **SHOW ALL YOUR WORK.** You will be graded on the methods you use as well as the accuracy of your answers. Correct answers without supporting work may not receive full credit.
- Write all work for each problem in the space provided. Be sure to write clearly and legibly. Erased or crossed out work will not be graded.
- Justifications require that you give mathematical (non-calculator) reasons and that you clearly identify functions, graphs, tables, or other objects that you use.
- You are permitted to use your calculator to solve an equation or graph a function without showing work. However, you must clearly indicate the setup of your problem.
- Your work must be expressed in mathematical notation rather than calculator syntax. For example,  $\int_1^5 x^2 dx$  may not be written as `fnInt(X^2,X,1,5)`.
- Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Good Luck!

NAME:



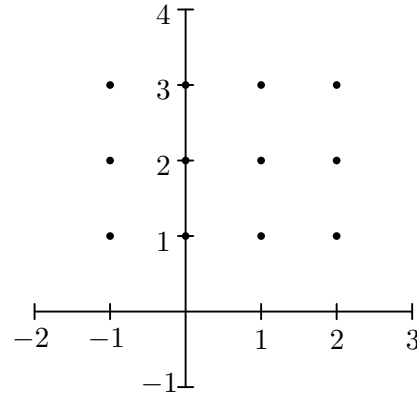
1. Let  $R$  be the region in the first quadrant enclosed by the graphs of  $y = 4e^{-x}$ ,  $y = \tan\left(\frac{x}{2}\right)$ , and the  $y$ -axis, as shown in the figure above.



- Find the area of region  $R$ .
- Find the volume of the solid generated when the region  $R$  is revolved about the  $x$ -axis.
- The region  $R$  is the base of a solid. For this solid, each cross-section perpendicular to the  $x$ -axis is a semicircle. Find the volume of this solid.

2. Consider the differential equation  $\frac{dy}{dx} = \frac{xy}{2}$  with initial condition  $y(0) = 2$ .

- Sketch the slope field for the given differential equation at the twelve points indicated.
- Sketch the solution curve that satisfies the initial condition  $y(0) = 2$  on the slope field above.
- Find the particular solution  $y = f(x)$  to the given differential equation with initial condition  $y(0) = 2$ . Then use your solution to find the exact value of  $y(2)$ .



3. A particle moves on the  $y$ -axis with velocity given by  $v(t) = t \sin(t^2)$  for  $t \geq 0$ .

- In which direction (up or down) is the particle moving at time  $t = 1.5$ ? Why?
- Find the acceleration of the particle at time  $t = 1.5$ . Is the velocity of the particle increasing at  $t = 1.5$ ? Why or why not?
- Given that  $y(t)$  is the position of the particle at time  $t$  and that  $y(0) = 3$ , find  $y(2)$ .
- Find the total distance traveled by the particle from  $t = 0$  to  $t = 2$ .



## CHAPTER 6

---

# TECHNIQUES of INTEGRATION

---

## 6.1 A Part, And Yet, Apart...

FIND ANTIDERIVATIVES OF THE FOLLOWING BY PARTS.

1243.  $\int x \ln x \, dx$

1248.  $\int \ln(4x) \, dx$

1244.  $\int \arctan x \, dx$

1249.  $\int 2^x x \, dx$

1245.  $\int 2xe^x \, dx$

1250.  $\int (x^2 - 5x)e^x \, dx$

1246.  $\int 3\theta \sin(2\theta) \, d\theta$

1251.  $\int e^x \sin x \, dx$

1247.  $\int \arcsin(2x) \, dx$

1252.  $\int x \sec^2 x \, dx$

SOLVE THE DIFFERENTIAL EQUATIONS.

1253.  $\frac{dy}{dx} = x^2 e^{4x}$

1255.  $\frac{dy}{d\theta} = \sin \sqrt{\theta}$

1254.  $\frac{dy}{dx} = x^2 \ln x$

1256.  $\frac{dy}{d\theta} = \theta \sec \theta \tan \theta$

SOLVE THE FOLLOWING.

1257. Find the area bounded by the curve  $y = \ln x$  and the lines  $y = 1$  and  $x = e^2$ .

1258. Find the area bounded by the curve  $y = \ln(x + 3)$ , the line  $y = 1$ , and the  $y$ -axis.

1259. Find the area of the region bounded entirely by the curves  $y = \ln x$  and  $y = (\ln x)^2$ .

1260. Find the area between the curves  $y = 5e^x$  and  $y = 4x^3 + \ln x$  over the interval  $[1, 2]$ .

1261. Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve  $y = e^x$ , and the line  $x = \ln 2$  about the line  $x = \ln 2$ .

1262. Find the average value of  $y = 2e^{-x} \cos x$  over the interval  $[0, 2\pi]$ .

1263. Graph the function  $f(x) = x \sin x$  in the window  $0 \leq x \leq 3\pi$ ,  $-5 \leq y \leq 10$ , using an  $x$ -scale of  $\pi$  and a  $y$ -scale of 5. Find the area of the region between  $f$  and the  $x$ -axis for

a)  $0 \leq x \leq \pi$

b)  $\pi \leq x \leq 2\pi$

c)  $2\pi \leq x \leq 3\pi$

d) What pattern do you see here? What is the area between the curve and the  $x$ -axis for  $n\pi \leq x \leq (n + 1)\pi$  for any nonnegative integer  $n$ ?

---

Advertising may be described as the science of arresting human intelligence long enough to get money from it. —*Stephen Leacock*

## 6.2 Partial Fractions

EVALUATE THE FOLLOWING BY PARTIAL FRACTIONS.

$$1264. \int \frac{1}{x^2 - 9} dx$$

$$1269. \int_{1/2}^1 \frac{x + 4}{x^2 + x} dx$$

$$1265. \int \frac{1}{1 - x^2} dx$$

$$1270. \int \frac{3x - 2}{x^3 - 3x^2 - 10x} dx$$

$$1266. \int \frac{3x - 2}{x^2 - 9} dx$$

$$1271. \int \frac{2}{x^2(x - 5)} dx$$

$$1267. \int \frac{x^2}{x^2 - 2x - 15} dx$$

$$1272. \int \frac{1}{x^2(x - 5)^2} dx$$

$$1268. \int \frac{3x^2 - 2x + 1}{9x^3 - x} dx$$

$$1273. \int_4^8 \frac{x}{x^2 - 2x - 3} dx$$

SOLVE THE FOLLOWING INITIAL VALUE PROBLEMS.

$$1274. \frac{dy}{dx} = (y^2 - y)e^x, \quad y(0) = 2$$

$$1276. \frac{dy}{dx} = \frac{1}{x^2 - 3x + 2}, \quad y(3) = 0$$

$$1275. \frac{dy}{d\theta} = (y + 1)^2 \sin \theta, \quad y(\pi/2) = 0$$

$$1277. \frac{dy}{dt} = \frac{2y + 2}{t^2 + 2t}, \quad y(1) = 1$$

**1278.** The growth of an animal population is governed by the equation  $1000 \frac{dP}{dt} = P(100 - P)$ , where  $P(t)$  is the number of animals in the colony at time  $t$ . If the initial population was 200 animals, how many animals will there be when  $t = 20$ ?

**1279.** Consider the equation  $\frac{dP}{dt} = 0.02P^2 - 0.08P$ . Sketch the slope field for this equation for  $0 \leq t \leq 50$  and  $0 \leq P \leq 8$ . Then sketch the solution curve corresponding to the initial condition  $P(0) = 1$ . Finally, solve the equation using the given initial condition.

**1280.** Sociologists sometimes use the phrase “social diffusion” to describe the way information spreads through a population, such as a rumor, cultural fad, or news concerning a technological innovation. In a sufficiently large population, the rate of diffusion is assumed to be proportional to the number of people  $p$  who have the information times the number of people who do not. Thus, if  $N$  is the population size, then  $\frac{dp}{dt} = kp(N - p)$ . Suppose that  $t$  is in days,  $k = \frac{1}{250}$ , and two people start a rumor at time  $t = 0$  in a population of  $N = 1000$  people. Find  $p(t)$  and determine how many days it will take for half the population to hear the rumor.

---

Biographical history, as taught in our public schools, is still largely a history of boneheads: ridiculous kings and queens, paranoid political leaders, compulsive voyagers, ignorant generals – the flotsam and jetsam of historical currents. The men who radically altered history, the great scientists and mathematicians, are seldom mentioned, if at all. —*Martin Gardner*

### 6.3 Trigonometric Substitution

EVALUATE THE FOLLOWING BY USING A TRIG SUBSTITUTION.

$$1281. \int \frac{3}{\sqrt{1+9x^2}} dx$$

$$1284. \int \frac{1}{x^2\sqrt{4x^2-9}} dx$$

$$1282. \int \frac{x^2}{\sqrt{9-x^2}} dx$$

$$1285. \int \frac{1}{(1-x^2)^{3/2}} dx$$

$$1283. \int \frac{\sqrt{x^2-9}}{x^2} dx$$

$$1286. \int \frac{\sqrt{4-x^2}}{x^2} dx$$

SOLVE THE FOLLOWING INITIAL VALUE PROBLEMS.

$$1287. x \frac{dy}{dx} = \sqrt{x^2-4}, \quad x \geq 2, \quad y(2) = 0$$

$$1289. (x^2+4) \frac{dy}{dx} = 3, \quad y(2) = 0$$

$$1288. \sqrt{x^2-9} \frac{dy}{dx} = 1, \quad x > 3, \quad y(5) = \ln 3$$

$$1290. (x^2+1)^2 \frac{dy}{dx} = \sqrt{x^2+1}, \quad y(0) = 1$$

SOLVE THE FOLLOWING PROBLEMS.

**1291.** Find the area of the region in the first quadrant that is enclosed by the coordinate axes and the curve  $y = \frac{1}{3}\sqrt{9-x^2}$ .

**1292.** Find the volume of the solid generated by revolving about the  $x$ -axis the region in the first quadrant bounded by the coordinate axes, the curve  $y = \frac{2}{1+x^2}$ , and the line  $x = 1$ .

**1293.** Consider the region bounded by the graphs of  $y = e^x$ ,  $y = 0$ ,  $x = 1$ , and  $x = 2$ . Find the value of  $d$  for which the line  $x = d$  divides the area of the region in a 2 : 1 ratio.

**1294.** Find the volume of the solids formed by revolving the following curves about the  $x$ -axis over the given interval.

a)  $y = xe^{x/2}$  over  $[0, 1]$

b)  $y = \sqrt{xe^x}$  over  $[1, 2]$

c)  $y = \ln x$  over  $[1, 2]$

d)  $y = \sqrt{1+x}$  over  $[1, 5]$

As I understand it, the first time Gabriel García Márquez opened Kafka's *The Metamorphosis*, he was a teenager, reclining on a couch. Upon reading

As Gregor Samsa awoke one morning from uneasy dreams he found himself transformed in his bed into a gigantic insect...

García Márquez fell off his couch, astonished by the revelation that you were *allowed* to write like that! It has happened to me often, and surely a similar thing happens to all mathematicians, that upon hearing of someone's new idea, or new construction, I have, like García Márquez, fallen off my (figurative) couch, thinking in amazement, "I didn't realize we were *allowed* to do that!" —Barry Mazur

## 6.4 Four Integral Problems

SOLVE EACH OF THE FOLLOWING.

**1295.** *Guess* which of the following two integrals will be larger. Explain your reasoning.

$$\int_0^4 x\sqrt{16-x^2} dx \quad \int_0^4 \sqrt{16-x^2} dx$$

Then *compute* which of the two integrals is actually larger.

**1296.** Show that the region enclosed by the graph of the parabola

$$f(x) = \frac{2}{a^2}x - \frac{1}{a^3}x^2, \quad a > 0$$

and the  $x$ -axis has an area that is independent of the value of  $a$ . How large is this area? What curve is determined by the vertices of all these parabolas?

**1297 (Calculator).** Let  $R$  be the region bounded by  $f(x) = e^{\sin 2x} \cos 2x$  and  $g(x) = x^2$ .

- a) Find the area of  $R$ . Your answer must include an antiderivative.
- b) Find the volume of the solid formed by revolving  $R$  about the line  $x = -1$ .
- c) Set up an integral that represents the volume of the solid whose base is  $R$  and the cross-sections perpendicular to the  $x$ -axis are squares. Use your calculator to evaluate the integral.

**1298.** Let  $f$  and  $g$  be continuous and differentiable functions satisfying the given conditions for some real number  $B$ :

- I.  $\int_1^3 f(x+2) dx = 3B$
- II. The average value of  $f$  in the interval  $[1, 3]$  is  $2B$
- III.  $\int_{-4}^x g(t) dt = f(x) + 3x$
- IV.  $g(x) = 4B + f'(x)$

a) Find  $\int_1^5 f(x) dx$  in terms of  $B$ .

b) Find  $B$ .

---

“Alice laughed: ‘There’s no use trying,’ she said; ‘one can’t believe impossible things.’  
 ‘I daresay you haven’t had much practice,’ said the Queen. ‘When I was younger, I always did it for half an hour a day. Why, sometimes I’ve believed as many as six impossible things before breakfast.’ ” —*Lewis Carroll*,  
 Through the Looking Glass

## 6.5 L'Hôpital's Rule

EVALUATE EACH OF THE FOLLOWING LIMITS.

$$1299. \lim_{x \rightarrow 0} \frac{e^x}{x}$$

$$1300. \lim_{x \rightarrow \infty} \frac{e^{4x}}{5x}$$

$$1301. \lim_{x \rightarrow \infty} \frac{(x+5)^2}{e^{3x}}$$

$$1302. \lim_{x \rightarrow 0} \frac{e^{3x} - 2^x}{3x}$$

$$1303. \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x$$

$$1304. \lim_{x \rightarrow \infty} 3x \ln \left(1 + \frac{1}{x}\right)$$

$$1305. \lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$$

$$1306. \lim_{x \rightarrow 2} \frac{\ln(3-x)}{1 - e^{x/2-1}}$$

$$1307. \lim_{x \rightarrow 0} \frac{\sin(3x)}{2x}$$

$$1308. \lim_{x \rightarrow 0} \frac{\sin(8x)}{6x}$$

$$1309. \lim_{x \rightarrow 0} \frac{\tan(3x)}{2x}$$

$$1310. \lim_{x \rightarrow \pi/2} \frac{1 - \sin x}{x - \pi/2}$$

$$1311. \lim_{x \rightarrow 1} \frac{3x^2 - 5x + 2}{x - 1}$$

$$1312. \lim_{x \rightarrow \infty} \frac{3x^2 - 5x + 2}{x - 1}$$

$$1313. \lim_{x \rightarrow 0^+} x^x$$

$$1314. \lim_{x \rightarrow \infty} (\ln x)^{1/x}$$

$$1315. \lim_{x \rightarrow \infty} (1 + 2x)^{1/(2 \ln x)}$$

$$1316. \lim_{x \rightarrow 1} (x^2 - 2x + 1)^{x-1}$$

$$1317. \lim_{x \rightarrow \infty} x^{1/x}$$

$$1318. \lim_{x \rightarrow 0^+} (1+x)^{1/x}$$

$$1319. \lim_{x \rightarrow 1} x^{1/(x-1)}$$

$$1320. \lim_{x \rightarrow 0^+} (\sin x)^x$$

$$1321. \lim_{x \rightarrow 0^+} (\sin x)^{\tan x}$$

$$1322. \lim_{x \rightarrow \infty} x^2 e^{-x}$$

$$1323. \lim_{x \rightarrow \infty} \int_x^{2x} \frac{1}{t} dt$$

$$1324. \lim_{x \rightarrow \infty} \frac{1}{x \ln x} \int_1^x \ln t dt$$

$$1325. \lim_{x \rightarrow 0} \frac{\cos x - 1}{e^x - x - 1}$$

$$1326. \lim_{x \rightarrow \infty} \frac{e^x + x^2}{e^x - x}$$

SOLVE THE FOLLOWING PROBLEMS.

1327. Find the value of  $c$  that makes the function below continuous at  $x = 0$ .

$$f(x) = \begin{cases} \frac{9x - 3 \sin(3x)}{5x^3} & x \neq 0 \\ c & x = 0 \end{cases}$$

1328. Estimate the value of  $\lim_{x \rightarrow 1} \frac{2x^2 - (3x+1)\sqrt{x} + 2}{x-1}$  by graphing. Then confirm your answer by using l'Hôpital's rule.

1329. Let  $f(x) = \frac{1 - \cos(x^6)}{x^{12}}$ .

- Graph  $f$  on the standard window ( $-10 \leq x \leq 10$ ,  $-10 \leq y \leq 10$ ) and use the graph to determine  $\lim_{x \rightarrow 0} f(x)$ .
- Now graph  $f$  on the window  $-1 \leq x \leq 1$ ,  $-0.5 \leq y \leq 1$ . What does the limit appear to be now?
- What does this indicate about finding limits using a graphing calculator?



## 6.6 Improper Integrals!

LET  $b$  BE A NUMBER GREATER THAN 1. EVALUATE THE FOLLOWING INTEGRALS IN TERMS OF  $b$ , THEN FIND THE LIMIT AS  $b \rightarrow \infty$ . WHAT DO THESE ANSWERS MEAN IN TERMS OF AREA UNDER THE CURVE?

$$1330. \int_1^b \frac{1}{x^2} dx$$

$$1332. \int_1^b \frac{1}{x^{1/2}} dx$$

$$1334. \int_1^b \frac{1}{x^{3/2}} dx$$

$$1331. \int_1^b \frac{1}{x^3} dx$$

$$1333. \int_1^b \frac{1}{x^{1/3}} dx$$

$$1335. \int_1^b \frac{1}{x} dx$$

LET  $b$  BE A NUMBER BETWEEN 0 AND 1. EVALUATE THE FOLLOWING INTEGRALS IN TERMS OF  $b$ , THEN FIND THE LIMIT AS  $b \rightarrow 0^+$ . WHAT DO THESE ANSWERS MEAN IN TERMS OF AREA UNDER THE CURVE?

$$1336. \int_b^1 \frac{1}{x^2} dx$$

$$1338. \int_b^1 \frac{1}{x^{1/2}} dx$$

$$1340. \int_b^1 \frac{1}{x^{3/2}} dx$$

$$1337. \int_b^1 \frac{1}{x^3} dx$$

$$1339. \int_b^1 \frac{1}{x^{1/3}} dx$$

$$1341. \int_b^1 \frac{1}{x} dx$$

EVALUATE THE FOLLOWING.

$$1342. \int_0^\infty \frac{1}{t^2 + 9} dt$$

$$1347. \int_0^{\pi/2} \tan \theta d\theta$$

$$1352. \int_0^1 \frac{4t}{\sqrt{1-t^4}} dt$$

$$1343. \int_0^\infty e^{-x} dx$$

$$1348. \int_e^\infty \frac{1}{x(\ln x)^2} dx$$

$$1353. \int_0^\infty \frac{16 \arctan x}{1+x^2} dx$$

$$1344. \int_{-\infty}^0 \frac{1}{(t-1)^2} dt$$

$$1349. \int_0^1 x \ln x dx$$

$$1354. \int_{-1}^\infty \frac{1}{x^2 + 5x + 6} dx$$

$$1345. \int_0^\infty \cos \theta d\theta$$

$$1350. \int_2^\infty \frac{1}{x\sqrt{x^2-1}} dx$$

$$1355. \int_1^\infty x^{-0.99} dx$$

$$1346. \int_0^\infty xe^{-x^2} dx$$

$$1351. \int_2^\infty \frac{2}{t^2-t} dt$$

$$1356. \int_1^\infty x^{-1.01} dx$$

**1357.** Consider the region in the first quadrant between the curve  $y = e^{-x}$  and the  $x$ -axis. Find the area of the region; the volume of the solid formed when the region is revolved about the  $y$ -axis; and the volume of the solid formed when the region is revolved about the  $x$ -axis.

**1358.** Let  $R$  be the region between the curves  $y = 1/x$  and  $y = 1/(x+1)$ , to the right of the line  $x = 1$ . Find the area of this region if it is finite.

**1359.** A patient is given an injection of imitrex, a medicine to relieve migraine headaches, at a rate of  $r(t) = 2te^{-2t}$  ml/sec, where  $t$  is the number of seconds since the injection started. Estimate the total quantity of imitrex injected.

## 6.7 The Art of Integration

**1360 (AP).** If the substitution  $u = x/2$  is made, the integral  $\int_2^4 \frac{1 - (x/2)^2}{x} dx =$

- A)  $\int_1^2 \frac{1 - u^2}{u} du$       B)  $\int_2^4 \frac{1 - u^2}{u} du$       C)  $\int_2^4 \frac{1 - u^2}{4u} du$   
 D)  $\int_2^4 \frac{1 - u^2}{2u} du$       E)  $\int_1^2 \frac{1 - u^2}{2u} du$

**1361.** *Partial Fractions Versus Trig Substitution*

- a) Graph the function  $f(x) = \frac{1}{x^2 - 4}$  on your paper.
- b) Is the definite integral  $\int_{-1}^1 \frac{dx}{x^2 - 4}$  negative or positive? Justify your answer with reference to your graph.
- c) Compute the integral in part (b) by using partial fractions.
- d) A Georgia Tech calculus student suggests instead to use the substitution  $x = 2 \sec \theta$ . Compute the integral in this way, or describe why this substitution fails.

**1362 (AP).** If  $\int f(x) \sin x dx = -f(x) \cos x + \int 3x^2 \cos x dx$ , then  $f(x)$  could be

- A)  $3x^2$       B)  $x^3$       C)  $-x^3$       D)  $\sin x$       E)  $\cos x$

**1363.** Justin and Jonathan are having an argument as to the value of  $\int \sec^2 x \tan x dx$ . Justin makes the substitution  $u = \sec x$  and gets the answer  $\frac{1}{2} \sec^2 x$ , whereas Jonathan makes the substitution  $u = \tan x$  and gets the answer  $\frac{1}{2} \tan^2 x$ . Please get them to stop arguing by explaining to them why their antiderivatives are both acceptable.

**1364.** Determine which of the following converge (are finite) and diverge (are infinite) by comparing the integrands to other known integrals.

- A)  $\int_1^\infty \frac{dx}{1 + x^4}$       B)  $\int_1^\infty \frac{x dx}{\sqrt{1 + x^3}}$       C)  $\int_0^\infty e^{-x^2} dx$       D)  $\int_1^\infty \frac{\sin x}{x^2} dx$

---

Only an idiot could believe that science requires martyrdom – that may be necessary in religion, but in time a scientific result will establish itself. —*David Hilbert*

**1365.** A function  $G$  is defined by  $G(x) = \int_0^x \sqrt{1+t^2} dt$  for all real numbers  $x$ . Determine whether the following statements are true or false. Justify your answers.

- A)  $G$  is continuous at  $x = 0$ .
- B)  $G(3) > G(1)$ .
- C)  $G'(2\sqrt{2}) = 3$ .
- D) The graph of  $G$  has a horizontal tangent at  $x = 0$ .
- E) The graph of  $G$  has an inflection point at  $(0, 0)$ .

**1366.** Consider the following table of values for the differentiable function  $f$ .

$x$	1.0	1.2	1.4	1.6	1.8
$f(x)$	5.0	3.5	2.6	2.0	1.5

- a) Estimate  $f'(1.4)$ .
- b) Give an equation for the tangent line to the graph of  $f$  at  $x = 1.4$ .
- c) Is  $f''(x)$  positive, negative, or zero? Explain how you determine this.
- d) Using the data in the table, find a midpoint approximation with 2 equal subintervals for  $\int_{1.0}^{1.8} f(x) dx$ .

**1367.** The gamma function  $\Gamma(x)$  is defined for all  $x > 0$  by

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt.$$

- a) Evaluate  $\Gamma(1)$ .
- b) For  $x > 1$ , show that  $\Gamma(x) = x\Gamma(x-1)$ . Assume that all these improper integrals exist. *Hint:* Use integration by parts.
- c) Use parts (a) and (b) to find  $\Gamma(2)$ ,  $\Gamma(3)$ , and  $\Gamma(4)$ . What is the pattern?
- d) One of the few values of  $\Gamma(x)$  for noninteger  $x$  that can be evaluated exactly is  $\Gamma(\frac{1}{2}) = \int_0^{\infty} e^{-t} t^{-1/2} dt$ , whose value is  $\sqrt{\pi}$ . Explain why  $\Gamma(\frac{1}{2})$  converges.
- e) Try to evaluate  $\Gamma(\frac{1}{2})$  on your calculator.

---

If you ask mathematicians what they do, you always get the same answer. They think. They think about difficult and unusual problems. They do not think about ordinary problems: they just write down the answers to those. —*M. Egrafov*

## 6.8 Functions Defined By Integrals

SOLVE THE FOLLOWING PROBLEMS.

**1368 (AP).**  $\lim_{x \rightarrow 0} \frac{\int_1^{1+x} \sqrt{t^2 + 8}}{x}$  is

- A) 0                      B) 1                      C) 3                      D)  $2\sqrt{2}$                       E) nonexistent

**1369.** Find the derivatives of the functions defined by the following integrals.

a)  $\int_0^x \frac{\sin t}{t} dt$

d)  $\int_0^1 e^{\tan^2 t} dt$

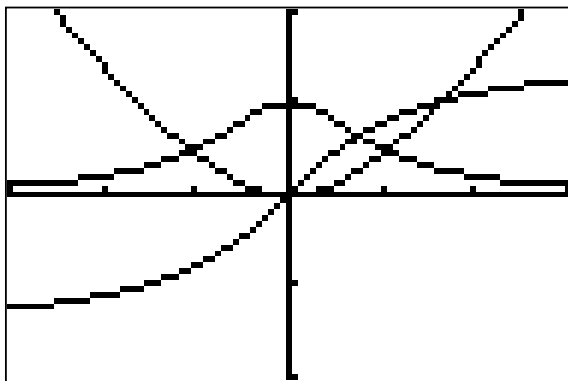
b)  $\int_0^x e^{-t^2} dt$

e)  $\int_1^{\ln x} e^{t^2} dt$

c)  $\int_1^{\cos t} \frac{1}{t} dt$

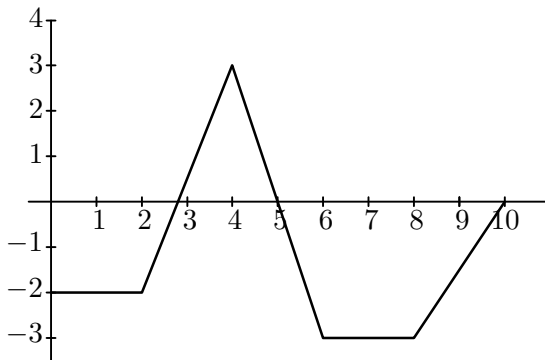
f)  $\int_x^{x^2} \frac{1}{2t} dt$

**1370.** The graphs of three functions appear in the figure below. Identify which is  $f(x)$ , which is  $f'(x)$ , and which is  $\int_0^x f(t) dt$ .

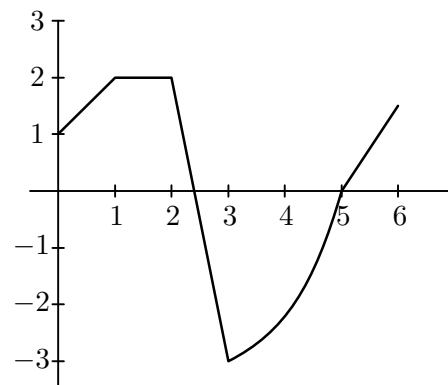


**1371.** Let  $F(x) = \int_0^x f(t) dt$  where  $f$  is the function graphed below.

- Evaluate  $\int_0^2 f(t) dt$ ,  $\int_0^4 f(t) dt$ ,  $\int_2^4 f(t) dt$ ,  $\int_5^{10} f(t) dt$ , and  $\int_1^7 f(t) dt$ .
- Evaluate  $F(0)$ ,  $F(2)$ ,  $F(5)$ , and  $F(7)$ .
- Find an analytic expression for  $f(x)$ .
- Find an analytic expression for  $F(x)$ .
- Sketch the graphs of  $f$  and  $F$  on the same axes over the interval  $[0, 10]$ .
- Where does  $F$  have local maxima on the interval  $[0, 10]$ ?
- On which subintervals of  $[0, 10]$ , if any, is  $F$  decreasing?
- On which subintervals of  $[0, 10]$ , if any, is  $F$  increasing?
- On which subintervals of  $[0, 10]$ , if any, is the graph of  $F$  concave up?
- On which subintervals of  $[0, 10]$ , if any, is the graph of  $F$  concave down?



Graph for #1371



Graph for #1372

**1372.** Let  $F(x) = \int_1^x f(t) dt$ , where  $f$  is the function graphed above.

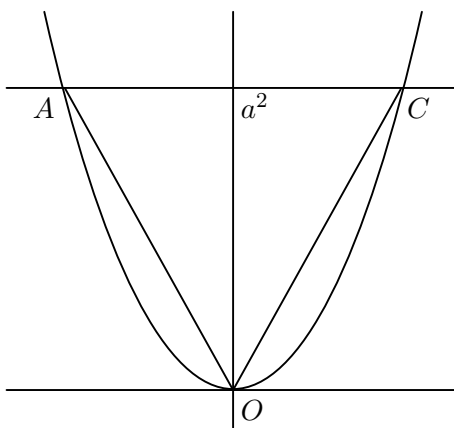
- Suppose  $\int_0^5 f(t) dt = -\frac{2}{3}$ . What is  $F(5)$ ?
- Show that  $F$  has exactly one zero between 3 and 4.
- Find the equation of the tangent line to the graph of  $F$  at the point  $(3, F(3))$ . *Hint:* What is  $F'(3)$ ?
- Use the equation found in part (c) to approximate the zero of  $F$  between 3 and 4.

## 6.9 Sample A.P. Problems on Techniques of Integration

**1373.** Let  $A(t)$  be the area of the region in the first quadrant enclosed by the coordinate axes, the curve  $y = e^{-x}$ , and the line  $x = t > 0$ . Let  $V(t)$  be the volume of the solid generated by revolving the region about the  $x$ -axis. Find the following limits.

$$\text{a) } \lim_{t \rightarrow \infty} A(t) \qquad \text{b) } \lim_{t \rightarrow \infty} \frac{V(t)}{A(t)} \qquad \text{c) } \lim_{t \rightarrow 0^+} \frac{V(t)}{A(t)}$$

**1374.** The figure below shows triangle  $AOC$  inscribed in the region cut from the parabola  $y = x^2$  by the line  $y = a^2$ . Find the limit of the ratio of the area of the triangle to the area of the parabolic region as  $a$  approaches zero.



**1375.** Find the area of the region enclosed by the curves  $y = x^2$ ,  $y = (x^2 + x + 1)e^{-x}$ , and the  $y$ -axis.

**1376.** Many chemical reactions are the result of the interaction of two molecules that undergo a change to produce a new product. The rate of the reaction typically depends on the concentration of the two kinds of molecules. If  $a$  is the amount of substance  $A$  and  $b$  is the amount of substance  $B$  at time  $t = 0$ , and if  $x$  is the amount of product at time  $t$ , then the rate of formation of the product may be given by the separable differential equation

$$\frac{dx}{dt} = k(a - x)(b - x)$$

where  $k$  is a constant for the reaction. Assuming that  $x = 0$  when  $t = 0$ , solve this equation to obtain a relation between  $x$  and  $t$ .

a) if  $a = b$  and

b) if  $a \neq b$ .

**1377.** For what value of  $a$  does

$$\int_1^{\infty} \left( \frac{ax}{x^2 + 1} - \frac{1}{2x} \right) dx$$

converge? To what value does it converge?

**1378.** Let  $R$  be the region in the first quadrant that is bounded above by the line  $y = 1$ , below by the curve  $y = \ln x$ , and on the left by the line  $x = 1$ . Find the volume of the solid generated by revolving the  $R$  about

- a) the  $x$ -axis
- b) the line  $y = 1$
- c) the  $y$ -axis
- d) the line  $x = 1$ .

**1379.** The region between the  $x$ -axis and the curve

$$y = \begin{cases} 0 & x = 0 \\ x \ln x & 0 < x \leq 2 \end{cases}$$

is revolved around the  $x$ -axis to generate a solid.

- a) Show that  $y$  is continuous at  $x = 0$ .
- b) Find the volume of the solid.

**1380.** A single infected individual enters a community of  $n$  susceptible individuals. Let  $x$  be the number of newly infected individuals at time  $t$ . The common epidemic model assumes that the disease spreads at a rate proportional to the product of the total number infected and the number not yet infected. Thus, the spread is modeled by

$$\frac{dx}{dt} = k(x+1)(n-x).$$

- a) Find  $x(t)$ , the solution to the differential equation, in terms of  $k$  and  $n$ .
- b) If an infected person enters a community of 1500 susceptible individuals, and 100 are infected after 15 days, how many days will it take for 1000 people to be infected?

**1381 (1996BC, Calculator).** Consider the graph of the function  $h$  given by  $h(x) = e^{-x^2}$  for  $0 \leq x < \infty$ .

- a) Let  $R$  be the unbounded region in the first quadrant below the graph of  $h$ . Find the volume of the solid generated when  $R$  is revolved about the  $y$ -axis.
- b) A rectangle has one vertex at the origin, one on the  $x$ -axis at  $x = w$ , one on the  $y$ -axis and another on the graph of  $h$  in the first quadrant. Let  $A(w)$  be the area of the rectangle. Show that  $A(w)$  has its maximum value when  $w$  is the  $x$ -coordinate of the point of inflection of the graph of  $h$ .

**1382.** Find the area under the arch of the ellipse  $y = \frac{4}{25}\sqrt{25 - x^2}$  and above the  $x$ -axis.

**1383 (1969BC).** A region  $R$  has  $y = 1 + \sin\left(\frac{\pi x}{2}\right)$  as its upper boundary,  $y = \frac{1}{2}x$  as its lower boundary, and the  $y$ -axis as its left-hand boundary.

- a) Sketch the region  $R$ .
- b) Set up, but do not evaluate, an integral expression in terms of the single variable  $x$ , for
  - i) the area of  $R$ ;
  - ii) the volume of the solid formed by revolving  $R$  about the  $x$ -axis;
  - iii) and the total perimeter of  $R$ .

**1384 (1980BC).** Let  $R$  be the region enclosed by the graphs of  $y = e^{-x}$ ,  $x = k$  ( $k > 0$ ), and the coordinate axes.

- a) Write an improper integral that represents the limit of the area of  $R$  as  $k$  increases without bound and find the value of the integral if it exists.
- b) Find the volume, in terms of  $k$ , of the solid generated by rotating  $R$  around the  $y$ -axis.
- c) Find the volume, in terms of  $k$ , of the solid whose base is  $R$  and whose cross sections perpendicular to the  $x$ -axis are squares.

**1385.** When computing the internal energy of a crystal, Claude Garrod, in his book *Twentieth Century Physics* (published in 1984), states that the integral

$$\int_0^{\pi/2} \frac{\sin x}{e^{0.26 \sin x} - 1} dx$$

“cannot be evaluated analytically. However, it can easily be computed numerically using Simpson’s rule. The result is 5.56.”

- a) Is the integral proper or improper? Why?
- b) What is the limit of the integrand as  $x \rightarrow 0^+$ ?
- c) What does “cannot be evaluated analytically” mean?
- d) Is it possible to use your calculator program to approximate the integral by Simpson’s rule with  $n = 6$ ? If so, approximate it to four decimal places; if not, why not?

---

More than 50% of all known mathematics was created during the past 50 years, and 50% of all mathematicians who have ever lived are alive today. —Howard Eves



## 6.10 Sample Multiple-Choice Problems on Techniques of Integration

**1386.**  $\int x \sin x \, dx =$

- A)  $-x \cos x + C$
- B)  $-x \cos x - \sin x + C$
- C)  $-x \cos x + \sin x + C$
- D)  $\frac{1}{2}x^2 \sin x + C$
- E)  $-x \cos x - \cos x + C$

**1387.**  $\int x e^{-x} \, dx =$

- A)  $e^{-x}(1-x) + C$
- B)  $\frac{e^{1-x}}{1-x} + C$
- C)  $-e^{-x}(x+1) + C$
- D)  $-\frac{1}{2}x e^{-x} + C$
- E)  $e^{-x}(x+1) + C$

**1388.**  $\int \frac{\ln x}{x} \, dx =$

- A)  $\frac{1}{2} \ln x + C$
- B)  $\frac{1}{2}(\ln x)^2 + C$
- C)  $2\sqrt{\ln x} + C$
- D)  $\frac{1}{2} \ln x^2 + C$
- E) None of these

**1392.**  $\int_0^1 \frac{e^x}{1+e^x} \, dx =$

- A)  $\ln 2$
- B)  $e$
- C)  $1+e$
- D)  $-\ln 2$
- E)  $\ln\left(\frac{e+1}{2}\right)$

**1393.**  $\int_0^{\pi/4} \tan^2 \theta \, d\theta =$

- A)  $\frac{\pi}{4} - 1$
- B)  $\sqrt{2} - 1$
- C)  $\frac{\pi}{4} + 1$
- D)  $\frac{1}{3}$
- E)  $1 - \frac{\pi}{4}$

**1389.**  $\int \tan^{-1}(2x) \, dx =$

- A)  $\frac{2}{1+4x^2} + C$
- B)  $x \tan^{-1}(2x) + C$
- C)  $x \tan^{-1}(2x) + \frac{1}{4} \ln(1+4x^2) + C$
- D)  $x \tan^{-1}(2x) - \frac{1}{4} \ln(1+4x^2) + C$
- E) None of these

**1390.**  $\int \frac{x}{\sqrt{9-x^2}} \, dx =$

- A)  $-\frac{1}{2} \ln \sqrt{9-x^2} + C$
- B)  $\arcsin\left(\frac{x}{3}\right) + C$
- C)  $-\sqrt{9-x^2} + C$
- D)  $-\frac{1}{4} \sqrt{9-x^2} + C$
- E)  $2\sqrt{9-x^2} + C$

**1391.**  $\int \tan x \, dx =$

- A)  $-\ln|\sec x| + C$
- B)  $\sec^2 x + C$
- C)  $\ln|\sin x| + C$
- D)  $\sec x + C$
- E)  $-\ln|\cos x| + C$

1394.  $\lim_{x \rightarrow \pi/2} \frac{\cos x}{x - \pi/2} =$   
 A)  $-1$       B)  $1$       C)  $0$       D)  $\infty$       E) None of these
1395.  $\int_0^1 x e^x dx =$   
 A)  $1$       B)  $-1$       C)  $2 - e$       D)  $\frac{1}{2}e^2 - e$       E)  $e - 1$
1396.  $\int_1^e \ln x dx =$   
 A)  $\frac{1}{2}$       B)  $e - 1$       C)  $e + 1$       D)  $1$       E)  $-1$
1397.  $\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right) =$   
 A)  $1$       B)  $0$       C)  $\infty$       D)  $-1$       E) None of these
1398. Which of the following integrals is equal to  $\frac{5}{4}$ ?  
 A)  $\int_0^1 \frac{1}{x^{0.2}} dx$     B)  $\int_0^1 \frac{1}{x^{0.5}} dx$     C)  $\int_0^1 \frac{1}{x^{0.7}} dx$     D)  $\int_0^1 \frac{1}{x^2} dx$     E) None of these
1399.  $\lim_{h \rightarrow 0} \frac{-1 + e^{-h}}{h} =$   
 A)  $1$       B)  $0$       C)  $-1$       D)  $\frac{1}{e}$       E)  $\infty$
1400. The region bounded by  $y = e^x$ ,  $y = 1$ , and  $x = 2$  is revolved about the  $y$ -axis. The volume of this solid is  
 A)  $2\pi(e^2 - 1)$     B)  $\pi(e^2 + 1)$     C)  $\pi(e^2 - 2)$     D)  $2\pi(e^2 - 2)$     E) None of these
1401. The region bounded by  $y = e^x$ ,  $y = 1$ , and  $x = 2$  is revolved about the  $x$ -axis. The volume of this solid is  
 A)  $\frac{\pi}{2}(e^4 - 4)$     B)  $\pi(e^4 - 4)$     C)  $\frac{\pi}{2}(e^4 - 5)$     D)  $\frac{\pi}{2}(e^4 - 10)$     E) None of these
1402. The area in the first quadrant bounded by the curve  $y = x^2$  and the line  $y - x - 2 = 0$  is  
 A)  $\frac{3}{2}$       B)  $\frac{2}{3}$       C)  $\frac{7}{6}$       D)  $\frac{10}{3}$       E)  $\frac{9}{2}$
1403.  $\frac{d}{dx} \int_3^{-5x^2} (7t - 1) dt =$   
 A)  $7x^2 - 1$     B)  $-5x^2 - 1$     C)  $-70x^3 + 10x$     D)  $350x^3 + 10x$     E) None of these

**A.P. Calculus BC Test Five**  
**Section One**  
**Multiple-Choice**  
**No Calculators**  
Time—35 minutes  
Number of Questions—15

The scoring for this section is determined by the formula

$$[C - (0.25 \times I)] \times 1.8$$

where  $C$  is the number of correct responses and  $I$  is the number of incorrect responses. An unanswered question earns zero points. The maximum possible points earned on this section is 27, which represents 50% of the total test score.

*Directions:* Solve each of the following problems, using the available space for scratch work. After examining the form of the choices, decide which is the best of the choices given and fill in the corresponding choice on your answer sheet. Do not spend too much time on any one problem.

Good Luck!

NAME:

1.  $\int x \sin x \, dx =$

- A)  $-x \cos x + C$
  - B)  $-x \cos x - \sin x + C$
  - C)  $-x \cos x + \sin x + C$
  - D)  $\frac{1}{2}x^2 \sin x + C$
  - E)  $-x \cos x - \cos x + C$
- 

2.  $\int_1^e \frac{\ln x}{x} \, dx =$

- A) undefined
  - B)  $\frac{1}{2}$
  - C) 2
  - D)  $\frac{1}{2}(e - 1)$
  - E) None of these
- 

3. The area of the region bounded by the lines  $x = 0$ ,  $x = 2$ ,  $y = 0$ , and the curve  $y = e^{x/2}$  is

- A)  $\frac{1}{2}(e - 1)$
- B)  $e - 1$
- C)  $2(e - 1)$
- D)  $2e - 1$
- E)  $2e$

4.  $\lim_{h \rightarrow 0} \frac{-1 + e^{-h}}{h} =$

- A) 1
- B) 0
- C) -1
- D)  $\frac{1}{e}$
- E)  $\infty$

---

5. Evaluate  $\int_1^{\infty} x^{-1/2} dx$ .

- A) 3
- B) 2
- C) 1
- D)  $\frac{1}{2}$
- E) divergent

---

6.  $\int \frac{1}{x^2 + x} dx =$

- A)  $\frac{1}{2} \arctan\left(x + \frac{1}{2}\right) + C$
- B)  $\ln|x^2 + x| + C$
- C)  $\ln\left|\frac{x+1}{x}\right| + C$
- D)  $\ln\left|\frac{x}{x+1}\right| + C$
- E) None of these

7.  $\int \frac{x}{x+2} dx =$

- A)  $x \ln |x+2| + C$
- B)  $x + 2 \ln |x+2| + C$
- C)  $x - 2 \ln |x+2| + C$
- D)  $x - \ln |x+2| + C$
- E)  $x - \arctan x + C$

---

8. A particle moves on the  $x$ -axis in such a way that its position at time  $t$ , for  $t > 0$ , is given by  $x(t) = (\ln t)^2$ . At what value of  $t$  does the velocity of the particle attain its maximum?

- A) 1
- B)  $e^{1/2}$
- C)  $e$
- D)  $e^{3/2}$
- E)  $e^2$

---

9. The substitution of  $x = \sin \theta$  in the integrand of  $\int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}} dx$  results in

- A)  $\int_0^{1/2} \frac{\sin^2 \theta}{\cos \theta} d\theta$
- B)  $\int_0^{1/2} \sin^2 \theta d\theta$
- C)  $\int_0^{\pi/6} \sin^2 \theta d\theta$
- D)  $\int_0^{\pi/3} \sin^2 \theta d\theta$
- E)  $\int_0^{1/2} \frac{\cos^2 \theta}{\sin \theta} d\theta$

10. The area of the region in the first quadrant under the curve  $y = \frac{1}{\sqrt{1-x^2}}$ , bounded on the left by  $x = \frac{1}{2}$ , and on the right by  $x = 1$  is

- A)  $\infty$
- B)  $\pi$
- C)  $\pi/2$
- D)  $\pi/3$
- E) None of these

---

11. The length of the curve  $y = \int_0^x \sqrt{\frac{t}{3}} dt$  from  $x = 0$  to  $x = 9$  is

- A) 16.
- B) 14.
- C)  $\frac{31}{3}$ .
- D)  $9\sqrt{3}$ .
- E)  $\frac{14}{3}$ .

---

12. Evaluate  $\int_{-5}^5 \sqrt{25-x^2} dx$ .

- A) 0
- B) 5
- C)  $25\pi/2$
- D)  $25\pi$
- E)  $50\pi$

13. Consider the function  $g$  defined by  $g(x) = \int_1^x (t^3 - 3t^2 + 2t) dt$ . The number of relative extrema of  $g$  is

- A) 1.
  - B) 2.
  - C) 3.
  - D) 4.
  - E) more than 4.
- 

14. The function  $t(x) = 2^x - \frac{|x-3|}{x-3}$  has

- A) a removable discontinuity at  $x = 3$ .
  - B) an infinite discontinuity at  $x = 3$ .
  - C) a jump discontinuity at  $x = 3$ .
  - D) no discontinuities.
  - E) a removable discontinuity at  $x = 0$  and an infinite discontinuity at  $x = 3$ .
- 

15. Find the values of  $c$  so that the function

$$h(x) = \begin{cases} c^2 - x^2 & x < 2 \\ x + c & x \geq 2 \end{cases}$$

is continuous everywhere.

- A)  $-3, -2$
- B)  $2, 3$
- C)  $-2, 3$
- D)  $-3, 2$
- E) There are no such values.



**A.P. Calculus BC Test Five**  
**Section Two**  
**Free-Response**  
**Calculators Allowed**  
Time—45 minutes  
Number of Questions—3

Each of the three questions is worth 9 points. The maximum possible points earned on this section is 27, which represents 50% of the total test score. There is no penalty for guessing.

- **SHOW ALL YOUR WORK.** You will be graded on the methods you use as well as the accuracy of your answers. Correct answers without supporting work may not receive full credit.
- Write all work for each problem in the space provided. Be sure to write clearly and legibly. Erased or crossed out work will not be graded.
- Justifications require that you give mathematical (non-calculator) reasons and that you clearly identify functions, graphs, tables, or other objects that you use.
- You are permitted to use your calculator to solve an equation or graph a function without showing work. However, you must clearly indicate the setup of your problem.
- Your work must be expressed in mathematical notation rather than calculator syntax. For example,  $\int_1^5 x^2 dx$  may not be written as `fnInt(X^2,X,1,5)`.
- Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Good Luck!

NAME:

1. Let  $f$  be a differentiable function defined for all  $x \geq 0$  such that  $f(0) = 5$  and  $f(3) = -1$ . Suppose that for any number  $b > 0$  the average value of  $f(x)$  on the interval  $0 \leq x \leq b$  is  $\frac{f(0) + f(b)}{2}$ .

a) Find  $\int_0^3 f(x) dx$ .

b) Prove that  $f'(x) = \frac{f(x) - 5}{x}$  for all  $x > 0$ .

c) Using part (b), find  $f(x)$ .

---

2. Let  $R$  be the region enclosed by the graph of  $y = \ln x$ , the line  $x = 3$ , and the  $x$ -axis.

a) Find the area of region  $R$  by evaluating an antiderivative.

b) Find the volume of the solid generated by revolving region  $R$  about the  $x$ -axis.

c) Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid generated by revolving the region  $R$  about the line  $x = 3$ .

---

3. Consider the differential equation given by  $\frac{dy}{dx} = \frac{-xy}{\ln y}$ .

a) Find the general solution of the differential equation.

b) Find the solution that satisfies the condition that  $y = e^2$  when  $x = 0$ . Express your answer in the form  $y = f(x)$ .

c) Explain why  $x = 2$  is not in the domain of the solution you found in part (b).

## CHAPTER 7

---

# **SERIES, VECTORS, PARAMETRICS and POLAR**

---

## 7.1 Sequences: Bounded and Unbounded

**1404.** An arithmetic sequence  $\{a_n\}$  has a first term  $a_1$  and a common difference  $d$ . Hence,  $\{a_n\} = a_1, a_1 + d, a_1 + 2d, a_1 + 3d, \dots = \{a_1 + (n - 1)d\}$ . Find an expression for the  $n$ th term of the following arithmetic sequences.

a)  $d = 4$  and  $a_1 = 7$                       b)  $d = 5$  and  $a_2 = 11$                       c)  $d = -\frac{1}{4}$  and  $a_{10} = \frac{1}{2}$

**1405.** A geometric sequence  $\{a_n\}$  has a first term  $a_1$  and a common ratio  $r$ . Hence,  $\{a_n\} = a_1, a_1r, a_1r^2, a_1r^3, \dots = \{a_1r^{n-1}\}$ . Find an expression for the  $n$ th term of the following geometric sequences.

a)  $r = \frac{1}{2}$  and  $a_1 = 2$                       b)  $r = \frac{1}{2}$  and  $a_1 = \frac{1}{4}$                       c)  $r = \frac{2}{3}$  and  $a_3 = 6$

### Limits of Sequences

$$\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0 \quad \lim_{n \rightarrow \infty} \sqrt[n]{n} = 1 \quad \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad \lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0$$

IF THE GIVEN SEQUENCE CONVERGES, FIND ITS LIMIT; OTHERWISE SAY THAT IT IS DIVERGENT.

**1406.**  $a_n = \frac{1 - 2n}{1 + 2n}$

**1410.**  $a_n = \frac{n}{2^n}$

**1415.**  $a_n = \frac{n!}{(2^n)(3^n)}$

**1407.**  $a_n = \frac{n + 3}{n^2 + 5n + 6}$

**1411.**  $a_n = \ln n - \ln(n + 1)$

**1416.**  $a_n = \left(\frac{3n + 1}{3n - 1}\right)^n$

**1408.**  $a_n = \left(\frac{n + 1}{2n}\right) \left(1 - \frac{1}{n}\right)$

**1412.**  $a_n = \sqrt[n]{10n}$

**1417.**  $a_n = \left(1 - \frac{1}{n^2}\right)^n$

**1409.**  $a_n = \sin\left(\frac{\pi}{2} + \frac{1}{n}\right)$

**1414.**  $a_n = \frac{n!}{n^n}$

**1418.**  $a_n = \arctan n$

**1419.**  $a_n = \sqrt[n]{n^2 + n}$

DETERMINE WHETHER THE FOLLOWING SEQUENCES ARE BOUNDED OR UNBOUNDED, AND WHETHER THEY ARE CONVERGENT OR DIVERGENT.

**1420.**  $a_n = \frac{(2n + 3)!}{(n + 1)!}$

**1423.**  $a_n = \frac{2^n - 1}{3^n}$

**1421.**  $a_n = 2 - \frac{2}{n} - \frac{1}{2^n}$

**1424.**  $a_n = \frac{n + 1}{n}$

**1422.**  $a_n = 1 - \frac{1}{n}$

**1425.**  $a_n = \frac{4^{n+1} + 3^n}{4^n}$

---

Mathematics is the cheapest science. Unlike physics or chemistry, it does not require any expensive equipment. All one needs for mathematics is a pencil and paper. —George Polya

## 7.2 It is a Question of Convergence...

DETERMINE WHETHER THE GIVEN SEQUENCE CONVERGES OR DIVERGES. IF IT CONVERGES, FIND ITS LIMIT.

$$1426. \left\{ \frac{n-2}{3n+2} \right\}$$

$$1430. \left\{ \frac{\log n}{2n} \right\}$$

$$1434. \left\{ \sqrt{n^2+2n} - 3n \right\}$$

$$1427. \left\{ \frac{n^3-1}{n} \right\}$$

$$1431. \left\{ \left(1 + \frac{1}{n}\right)^n \right\}$$

$$1435. \left\{ \frac{\ln e^n}{e^{2n}} \right\}$$

$$1428. \left\{ \frac{e^{3n}}{n} \right\}$$

$$1432. \{ \sin(n\pi) \}$$

$$1436. \left\{ \ln \left(1 + \frac{5}{n}\right)^n \right\}$$

$$1429. \left\{ \frac{n^3}{\ln(n+1)} \right\}$$

$$1433. \left\{ \cos\left(\frac{n\pi}{4}\right) \right\}$$

$$1437. \left\{ \frac{2n}{e^n} \right\}$$

CLASSIFY EACH OF THE FOLLOWING AS INCREASING, DECREASING, OR OSCILLATING.

$$1438. \left\{ \frac{2n+5}{n+7} \right\}$$

$$1441. \left\{ \frac{n!}{e^n} \right\}$$

$$1444. \{ ne^{-3n} \}$$

$$1439. \left\{ 2n - \frac{2}{n} \right\}$$

$$1442. \left\{ \frac{3^n}{2n!} \right\}$$

$$1445. \left\{ \frac{3^n}{1+3^n} \right\}$$

$$1440. \left\{ 2n + \frac{2}{n} \right\}$$

$$1443. \left\{ \sin\left(\frac{n\pi}{6}\right) \right\}$$

$$1446. \{ \arctan(2n) \}$$

1447. Let  $\{a_n\}$  be the sequence defined below.

$$a_n = \begin{cases} \frac{n^2}{n^2-20} & \text{if } n \text{ is a multiple of } 3 \\ \frac{n+1}{n} & \text{if } n \text{ is one more than a multiple of } 3 \\ \frac{\sqrt{n}}{\sqrt{n+3}} & \text{if } n \text{ is two more than a multiple of } 3 \end{cases}$$

Evaluate  $\lim_{n \rightarrow \infty} a_n$  or show that it does not exist.

1448. Which of the following are always true, and which are at least sometimes false?

- |  |  |
|--|--|
| <p>a) If <math>a_n &gt; 0</math> for all <math>n</math> and <math>a_n \rightarrow L</math>, then <math>L &gt; 0</math>.</p> <p>b) If <math>a_n \geq 0</math> for all <math>n</math> and <math>a_n \rightarrow L</math>, then <math>L \geq 0</math>.</p> <p>c) If <math>\{a_n\}</math> is bounded, then it converges.</p> <p>d) If <math>\{a_n\}</math> is not bounded, then it diverges.</p> | <p>e) If <math>\{a_n\}</math> is decreasing, then it converges.</p> <p>f) If <math>\{a_n\}</math> is decreasing and <math>a_n &gt; 0</math> for all <math>n</math>, then it converges.</p> <p>g) If <math>\{a_n\}</math> is neither increasing nor decreasing, then it diverges.</p> <p>h) If <math>a_n &gt; 0</math> for all <math>n</math> and <math>(-1)^n a_n \rightarrow L</math>, then <math>L = 0</math>.</p> |
|--|--|

### 7.3 Infinite Sums

EVALUATE THE SUM OF EACH SERIES.

$$1449. \sum_{n=1}^{\infty} \frac{1}{2^n}$$

$$1452. \sum_{n=1}^{\infty} \left(-\frac{1}{4}\right)^n$$

$$1455. \sum_{n=0}^{\infty} \left(\frac{1}{2^n} - \frac{(-1)^n}{5^n}\right)$$

$$1450. \sum_{n=1}^{\infty} (0.6)^n$$

$$1453. \sum_{n=1}^{\infty} \left(\frac{1}{2^n} - \frac{1}{3^n}\right)$$

$$1456. \sum_{n=0}^{\infty} \frac{2^{n+1}}{5^n}$$

$$1451. \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$$

$$1454. \sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n}\right)$$

$$1457. \sum_{n=1}^{\infty} \frac{3^{n-1} - 1}{6^{n-1}}$$

FOR THE FOLLOWING GEOMETRIC SERIES, FIND THE VALUES OF  $x$  FOR WHICH THE SERIES HAS A SUM.

$$1458. \sum_{n=0}^{\infty} (-1)^n x^n$$

$$1462. \sum_{n=0}^{\infty} 2^n x^n$$

$$1459. \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$1463. \sum_{n=0}^{\infty} (-1)^n x^{-2n}$$

$$1460. \sum_{n=0}^{\infty} 3 \left(\frac{x-1}{2}\right)^n$$

$$1464. \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-3)^n$$

$$1461. \sum_{n=0}^{\infty} \frac{(-1)^n}{2} \left(\frac{1}{3 + \sin x}\right)^n$$

$$1465. \sum_{n=0}^{\infty} (\ln x)^n$$

EXPRESS THE REPEATING DECIMALS AS FRACTIONS.

$$1466. 0.2323232323\dots$$

$$1468. 0.7777777777\dots$$

$$1470. 1.24123123123\dots$$

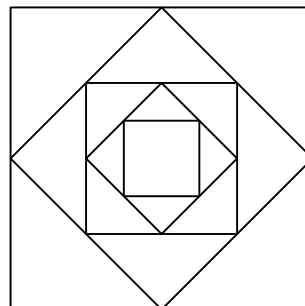
$$1467. 0.234234234\dots$$

$$1469. 1.4141414141\dots$$

$$1471. 3.142857142857\dots$$

**1472.** A ball is dropped from a height of 4 meters. Each time it strikes the pavement after falling from a height of  $h$  meters, it rebounds to a height of  $\frac{3}{4}h$  meters. Find the total distance the ball travels up and down.

**1473.** The figure shows the first five of a sequence of squares. The outermost square has an area of 4 square meters. Each of the other squares is obtained by joining the midpoints of the sides of the squares before it. Find the sum of the areas of all the squares.



## 7.4 Tests for Convergence and Divergence

IF THE SERIES CONVERGES, FIND ITS SUM; OTHERWISE GIVE A REASON WHY IT DIVERGES.

$$1474. \sum_{n=0}^{\infty} (\sqrt{2})^n$$

$$1477. \sum_{n=1}^{\infty} \ln \frac{1}{n}$$

$$1475. \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{2^n}$$

$$1478. \sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n$$

$$1476. \sum_{n=0}^{\infty} \frac{\cos(n\pi)}{5^n}$$

$$1479. \sum_{n=0}^{\infty} \frac{e^{n\pi}}{\pi^{ne}}$$

DETERMINE THE CONVERGENCE OR DIVERGENCE OF THE FOLLOWING SERIES BY USING AN APPROPRIATE TEST: THE *n*th term test, THE *integral test*, THE *p*-series test, THE *direct comparison test*, THE *limit comparison test*, THE *ratio test*, OR THE *root test*.

$$1480. \sum_{n=2}^{\infty} \frac{e^n}{1 + e^{2n}}$$

$$1487. \sum_{n=1}^{\infty} \left(\frac{n-2}{n}\right)^n$$

$$1481. \sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$$

$$1488. \sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$$

$$1482. \sum_{n=1}^{\infty} \frac{1}{1 + \ln n}$$

$$1489. \sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$$

$$1483. \sum_{n=1}^{\infty} \frac{n!}{e^n}$$

$$1490. \sum_{n=1}^{\infty} \frac{2 + (-1)^n}{1.25^n}$$

$$1484. \sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$$

$$1491. \sum_{n=1}^{\infty} \frac{\ln n}{n}$$

$$1485. \sum_{n=1}^{\infty} \frac{n}{n+1}$$

$$1492. \sum_{n=1}^{\infty} \frac{\arctan n}{n^{1.1}}$$

$$1486. \sum_{n=1}^{\infty} \frac{1}{(1 + \ln n)^2}$$

$$1493. \sum_{n=1}^{\infty} \frac{2}{1 + e^n}$$

1494. For what values of  $a$ , if any, does the series below converge?

$$\sum_{n=1}^{\infty} \left( \frac{a}{n+2} - \frac{1}{n+4} \right)$$

## 7.5 More Questions of Convergence...

WHICH OF THE SERIES BELOW CONVERGE ABSOLUTELY, WHICH CONVERGE CONDITIONALLY, AND WHICH DIVERGE?

$$1495. \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(0.1)^n}{n}$$

$$1501. \sum_{n=1}^{\infty} (-5)^{-n}$$

$$1507. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{5^n}$$

$$1496. \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^3 + 1}$$

$$1502. \sum_{n=1}^{\infty} \frac{\cos n\pi}{n\sqrt{n}}$$

$$1508. \sum_{n=1}^{\infty} \frac{(-3)^n}{(n+1)!}$$

$$1497. \sum_{n=1}^{\infty} (-1)^n \frac{\sin n}{n^2}$$

$$1503. \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(n!)^2}{(2n)!}$$

$$1509. \sum_{n=1}^{\infty} \frac{(-1)^n (n+2)!}{e^n}$$

$$1498. \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1+n}{n^2}$$

$$1504. \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2n)!}{2^n n! n}$$

$$1510. \sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}$$

$$1499. \sum_{n=1}^{\infty} (-1)^{n+1} \sqrt[n]{10}$$

$$1505. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n} + \sqrt{n+1}}$$

$$1511. \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n+1}$$

$$1500. \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n+1}$$

$$1506. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\left(1 + \frac{1}{n}\right)^n}$$

$$1512. \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{4n}$$

ESTIMATE THE ERROR IN USING THE FIRST FOUR TERMS TO APPROXIMATE THE SUM.

$$1513. \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$$

$$1515. \sum_{n=1}^{\infty} (-1)^n \frac{1}{n!}$$

$$1517. \sum_{n=1}^{\infty} (-1)^n \frac{n}{100^n}$$

$$1514. \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{10^n}$$

$$1516. \sum_{n=1}^{\infty} \left(\frac{-1}{4}\right)^n$$

$$1518. \sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n)!}$$

---

Even fairly good students, when they have obtained the solution of the problem and written down neatly the argument, shut their books and look for something else. Doing so, they miss an important and instructive phase of the work. ... A good teacher should understand and impress on his students the view that no problem whatever is completely exhausted.

One of the first and foremost duties of the teacher is not to give his students the impression that mathematical problems have little connection with each other, and no connection at all with anything else. We have a natural opportunity to investigate the connections of a problem when looking back at its solution. —George Polya



## 7.6 Power Series!

FIND THE INTERVAL OF CONVERGENCE OF EACH OF THE POWER SERIES BELOW.

$$1519. \sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n}$$

$$1525. \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n x^n$$

$$1531. \sum_{n=1}^{\infty} \frac{4^n x^n}{n}$$

$$1520. \sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{n}$$

$$1526. \sum_{n=0}^{\infty} n! (x-4)^n$$

$$1532. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (x-3)^n}{\sqrt{n}}$$

$$1521. \sum_{n=1}^{\infty} \frac{x^n}{3^{n^2} \sqrt{n}}$$

$$1527. \sum_{n=2}^{\infty} \frac{x^n}{n(\ln n)^2}$$

$$1533. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (x-3)^n}{5^n \sqrt{n}}$$

$$1522. \sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$$

$$1528. \sum_{n=1}^{\infty} \frac{(3x-1)^{n+1}}{2n+2}$$

$$1534. \sum_{n=1}^{\infty} \frac{(ex)^n}{\ln(n+1)}$$

$$1523. \sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n^2+3}}$$

$$1529. \sum_{n=0}^{\infty} \frac{x^n}{2n+1}$$

$$1535. \sum_{n=1}^{\infty} \frac{n^n x^n}{e^n}$$

$$1524. \sum_{n=0}^{\infty} \frac{nx^n}{4^n (n^2+1)}$$

$$1530. \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2+2}$$

$$1536. \sum_{n=1}^{\infty} \frac{n^n x^n}{(n+1)!}$$

FIND THE SERIES' INTERVAL OF CONVERGENCE AND, WITHIN THAT INTERVAL, THE SUM OF THE SERIES AS A FUNCTION OF  $x$ .

$$1537. \sum_{n=0}^{\infty} \frac{(x-1)^{2n}}{4^n}$$

$$1538. \sum_{n=0}^{\infty} (\ln x)^n$$

$$1539. \sum_{n=0}^{\infty} \left(\frac{x^2-1}{2}\right)^n$$

**1540.** The sum of a geometric series with  $|x| < 1$  can be viewed as a power series that represents the function  $\frac{1}{1-x}$ . In other words, the series

$$1 + x + x^2 + x^3 + x^4 + x^5 + \cdots = \frac{1}{1-x}$$

converges to  $\frac{1}{1-x}$  for all  $x$  in the interval  $-1 < x < 1$ .

- Differentiate the series term-by-term to find a power series representation for the function  $\frac{-1}{(1-x)^2}$ .
- Integrate the series term-by-term to find a power series representation for the function  $-\ln(1-x) = \ln \frac{1}{1-x}$ .
- Find a power series for the function  $\frac{1}{1+x}$ .

## 7.7 Maclaurin Series

*Table of Useful Maclaurin Series*

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \dots$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - \dots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots$$

**1541.** Find the intervals of convergence for each of the Maclaurin series in the table above.

**1542.** Using variable substitution, identities, differentiation, or integration on the series in the table above, find series representations for each of the following functions.

a)  $\frac{1}{1+x^2}$

c)  $xe^x$

e)  $\sin^2 x$

g)  $e^{x^2}$

b)  $\sin 2x$

d)  $\cos^2 x$

f)  $\frac{x^2}{1-2x}$

h)  $\int e^{x^2} dx$

**1543.** Use your calculator to find the values of each of the functions in the table above at  $x = \frac{1}{2}$ . Then, using the first 5 terms in each series, use your calculator to determine the error in approximating each of the functions in the table above at  $x = \frac{1}{2}$ .

**1544.** Let  $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0. \end{cases}$  With this definition of  $f$ , we have that the  $k$ th derivative of  $f$  exists for all positive integer  $k$ . In fact,  $f^{(k)}(0) = 0$ .

a) What is the Maclaurin series for  $f$ ? What is the interval of convergence?

b) Show that  $f'(0) = 0$  by evaluating  $\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x}$ .

**1545.** Why doesn't  $f(x) = x^{1/3}$  have a Maclaurin series?

## 7.8 Taylor Series

FOR THE FOLLOWING PROBLEMS, A) FIND THE TAYLOR SERIES EXPANSION FOR  $f$  ABOUT  $x = a$ ; B) EXPRESS THE SERIES IN SIGMA NOTATION; AND C) FIND THE INTERVAL OF CONVERGENCE.

**1546.**  $f(x) = \frac{1}{x} \quad a = 3$

**1550.**  $f(x) = \sin(\pi x) \quad a = \frac{1}{2}$

**1547.**  $f(x) = \cos x \quad a = \frac{\pi}{2}$

**1551.**  $f(x) = \frac{1}{x^2} \quad a = 1$

**1548.**  $f(x) = \frac{1}{x+2} \quad a = 3$

**1552.**  $f(x) = e^x \quad a = 2$

**1549.**  $f(x) = \ln x \quad a = 1$

**1553.**  $f(x) = 2^x \quad a = 1$

**1554.** For each of the above problems, use the Lagrange Error Bound to determine the maximum error in using only the first 3 terms of each series to approximate  $f$  at  $x = 3/2$ .

**1555.** Find the Maclaurin series for  $F(x) = \sqrt{1+x}$ .

**1556.** Show that  $\sum_{k=0}^{\infty} \frac{1}{k!}$  converges.

**1557.** Determine the convergence or divergence of  $\sum_{n=1}^{\infty} 2^{-1/n}$ .

**1558.** Let  $f$  be a function that has derivatives of all orders for all real numbers. Assume  $f(0) = 9$ ,  $f'(0) = 5$ ,  $f''(0) = -4$ , and  $f'''(0) = 36$ . Write the third order Taylor polynomial for  $f$  centered at  $x = 0$  and use it to approximate  $f(0.3)$ .

**1559.** Find the Taylor polynomial of order 3 generated by  $f(x) = \cos 3x$  centered at  $x = \frac{\pi}{3}$ .

**1560.** Find the Taylor polynomial of order 3 generated by  $f(x) = \sin 2x$  centered at  $x = \frac{\pi}{4}$ .

**1561. The Binomial Series** If we define  $\binom{m}{k}$  as

$$\binom{m}{1} = m, \quad \binom{m}{2} = \frac{m(m-1)}{2}, \quad \binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}, \quad k \geq 3$$

then we have the Maclaurin series representation

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k.$$

- a) What is the interval of convergence?
- b) What are the first four terms in the series representation for  $\sqrt{1+x}$ ?
- c) The first two terms are used to approximate  $\sqrt{1+x}$  when  $x$  is small. Is this enough terms to obtain an approximation accurate to two decimal places?

## 7.9 Vector Basics

USE THE FOLLOWING VECTORS TO ANSWER THE QUESTIONS BELOW:  $\mathbf{u} = \langle 3, -2 \rangle$ ,  $\mathbf{v} = \langle -5, 3 \rangle$ , AND  $\mathbf{w} = \langle 7, -1 \rangle$ .

**1562.** Find the magnitude  $\|\mathbf{u}\|$  of the vector  $\mathbf{u}$ . Find  $\|\mathbf{v}\|$  and  $\|\mathbf{w}\|$ .

**1563.** Evaluate  $5\mathbf{u} - \mathbf{v}$  and  $-2\mathbf{u} + 3\mathbf{w}$ .

**1564.** Find  $\mathbf{u} \cdot \mathbf{v}$ ;  $\mathbf{u} \cdot \mathbf{w}$ ;  $\mathbf{v} \cdot \mathbf{w}$ .

**1565.** Find the cosine of the angle between  $\mathbf{v}$  and  $\mathbf{w}$ .

**1566.** Find a unit vector in the direction of  $\mathbf{v}$ ; of  $\mathbf{w}$ .

**1567.** Find the area of the parallelogram with adjacent sides  $\mathbf{u}$  and  $\mathbf{v}$ .

**1568.** Evaluate  $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$ .

**1569.** Evaluate  $\mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ .

**1570.** Find the magnitude of the vector from  $P(2, 4)$  to  $Q(8, 1)$ .

**1571.** Find the magnitude of the vector  $2\mathbf{i} - 9\mathbf{j}$ .

**1572.** Which of the following vectors are orthogonal?  $2\mathbf{i} - 3\mathbf{j}$ ,  $6\mathbf{i} + 8\mathbf{j}$ ,  $-4\mathbf{i} + 3\mathbf{j}$ ,  $15\mathbf{i} + 10\mathbf{j}$

**1573.** If  $\|\mathbf{a}\| = 0$ , then what can you conclude about the vector  $\mathbf{a}$ ?

ANSWER THE FOLLOWING USING THE VECTOR-VALUED FUNCTIONS  $\mathbf{F}(t) = t^2\mathbf{i} - 2t\mathbf{j}$  AND  $\mathbf{G}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$ .

**1574.** Find  $\mathbf{G}(0)$  and  $\mathbf{G}(\frac{\pi}{3})$ .

**1575.** Find an expression for  $\|\mathbf{F}\|$  and  $\|\mathbf{G}\|$  for any  $t$ .

**1576.** For each value of  $t$ , the vector  $\mathbf{F}(t)$  can be interpreted as a point in the  $xy$ -plane. For example, when  $t = -1$ , we have  $\mathbf{F}(-1) = \mathbf{i} + 2\mathbf{j}$ ; this can be plotted as the point  $(1, 2)$ . Plot the points determined by  $\mathbf{F}$  for  $t = 0, 1, 2, 3, 4, 5$ . Connect the points with a continuous curve. What shape does the graph appear to have?

**1577.** Plot the points determined by  $\mathbf{G}$  for  $t = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$ . Connect the points with a continuous curve. What shape does the graph appear to have?

**1578.** Are  $\mathbf{F}(t)$  and  $\mathbf{G}(t)$  ever orthogonal for any  $t$ ? Justify your answer.

**1579.** What is the relationship between a parametrically defined function  $x = x(t)$ ,  $y = y(t)$  and the vector-valued function  $\mathbf{F}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ ?

## 7.10 Calculus with Vectors and Parametrics

FIND THE EQUATION OF THE TANGENT LINE TO THE CURVE AT THE GIVEN POINT.

**1580.**  $x = 2 \cos t, y = 2 \sin t, \quad t = \frac{\pi}{4}$

**1581.**  $x = -\sqrt{t+1}, y = \sqrt{3t}, \quad t = 3$ . Over what values of  $t$  is this curve defined?

**1582.**  $\mathbf{F}(t) = (\cos t)\mathbf{i} + (1 + \sin t)\mathbf{j}, \quad t = \frac{\pi}{2}$

**1583.**  $\mathbf{F}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}, \quad t = \frac{\pi}{3}$

**1584.** Find the second derivative for each of the above four curves.

GIVEN THE POSITION OF A PARTICLE, FIND A) THE VELOCITY VECTOR FOR ANY  $t$ , B) THE ACCELERATION VECTOR FOR ANY  $t$ , C) THE VELOCITY VECTOR AT THE GIVEN VALUE OF  $t$ , AND D) THE SPEED OF THE PARTICLE AT THE GIVEN VALUE OF  $t$ .

**1585.**  $x = t^2, y = 3t^3, \quad t = 1$

**1588.**  $\mathbf{R}(t) = (\sin \pi t)\mathbf{i} - (\cos \pi t)\mathbf{j}, \quad t = 3$

**1586.**  $x = \ln t, y = t^2, \quad t = e$

**1589.**  $\mathbf{R}(t) = (e^{2t})\mathbf{i} + \frac{t^2}{\ln 2}\mathbf{j}, \quad t = \ln 2$

**1587.**  $x = 2 \sin t, y = -3 \cos t, \quad t = \frac{\pi}{4}$

**1590.**  $\mathbf{R}(t) = [\tan(\frac{\pi t}{4})]\mathbf{i} - [\ln(t-2)]\mathbf{j}, \quad t = 3$ .

**1591.** Find  $\int_1^2 [(6-6t)\mathbf{i} + 3\sqrt{t}\mathbf{j}] dt$ .

**1592.** Find  $\int_2^4 [\frac{1}{t}\mathbf{i} + \frac{1}{5-t}\mathbf{j}] dt$ .

SOLVE THE INITIAL-VALUE PROBLEMS.

**1593.**  $\frac{d\mathbf{R}}{dt} = \frac{3}{2}\sqrt{t+1}\mathbf{i} + e^{-t}\mathbf{j}, \quad \mathbf{R}(0) = \mathbf{0}$

**1594.**  $\frac{d^2\mathbf{R}}{dt^2} = -32\mathbf{j}, \quad \mathbf{R}(0) = 100\mathbf{i}, \quad \mathbf{R}'(0) = 8\mathbf{i} + 8\mathbf{j}$

FIND THE ARC LENGTH OF EACH CURVE OVER THE INTERVAL GIVEN.

**1595.**  $x = 3t - 2, y = 2t + 1, \quad 0 \leq t \leq 3$

**1596.**  $x = 3 \cos t, y = 3 \sin t, \quad 0 \leq t \leq \pi$

**1597.**  $x = 3(2 + \frac{1}{3}t)^2, y = (2 + \frac{1}{3}t)^2, \quad 0 \leq t \leq 1$

**1598.**  $\mathbf{F}(t) = -2t\mathbf{i} + (3t - 1)\mathbf{j}, \quad -2 \leq t \leq 2$

---

When you teach the kids, tell them the truth. Tell them nothing but the truth. But, for God's sake, don't tell them the whole truth. —Cal Moore

## 7.11 Vector-Valued Functions

**1599.** A particle moves in the  $xy$ -plane so that at any time  $t > 0$ , its position  $(x, y)$  is given by  $x = 3 \cos t$  and  $y = 2 \sin t$ .

- Find the velocity vector at any time  $t$ .
- Find the acceleration vector at any time  $t$ .
- Find  $\lim_{t \rightarrow \pi/4} \frac{dy/dt}{dx/dt}$ .
- The particle moves on an ellipse. Find the equation of the ellipse without  $t$ .
- Find the equation of the tangent line to the ellipse at  $t = \frac{\pi}{4}$ .

**1600.** A particle moves in the  $xy$ -plane so that at any time  $t$ , its position is given by  $\mathbf{R}(t) = \langle t, 9t^2 - 4 \rangle$ . Let  $P$  represent the path that the particle is tracing.

- Find  $\mathbf{R}'(t)$  and  $\mathbf{R}''(t)$  at any time  $t$ .
- Sketch  $P$ .
- Find the equation of the tangent line to  $P$  at  $t = 1$ .

**1601.** Given the parametric equations  $x = 3(t - \cos t)$  and  $y = 2(3 - \sin t)$ ,

- find  $\frac{dy}{dx}$  at  $t = \frac{\pi}{3}$ ;
- find the equation of the tangent line to the graph at  $t = \frac{\pi}{3}$ ; and
- find the equation of the normal line to the graph at  $t = \frac{\pi}{3}$ .

**1602 (AP).** A particle moves on the circle  $x^2 + y^2 = 1$  so that at time  $t \geq 0$  its position  $(x, y)$  is given by the vector  $\left\langle \frac{1-t^2}{1+2t^2}, \frac{2t}{1+t^2} \right\rangle$ .

- Find the velocity vector.
- Is the particle ever at rest? Justify your answer.
- Give the coordinates of the point that the particle approaches as  $t$  increases without bound.

**1603.** A particle is moving along the curve described by the set of parametric equations  $x = t$  and  $y = \ln(\sec t)$ .

- Find an expression for the velocity vector at any time  $t$ .
- Find an expression for the acceleration vector at any time  $t$ .
- Find the magnitude of the velocity vector at  $t = \frac{\pi}{4}$ .
- Find the magnitude of the acceleration vector at  $t = \frac{\pi}{4}$ .

## 7.12 Motion Problems with Vectors

**1604.** The position vector of a particle is  $\mathbf{R}(t) = (3t + 1)\mathbf{i} + t^2\mathbf{j}$ . Find the angle between the velocity vector and the acceleration vector when  $t = 2$ .

**1605.** The position of a particle is given by the vector  $\langle e^{3t^2-1}, \ln(t^2 + 2t^4) \rangle$  where  $t > 0$ .

- a) Find the velocity vector.
- b) Evaluate the velocity at  $t = 1$ .
- c) Determine whether the particle is ever at rest. Explain.
- d) Find the coordinates of the point that the particle approaches as  $t$  approaches 1.

**1606.** The position of a particle is given by  $\mathbf{R}(t) = (\sin t)\mathbf{i} + (\cos 2t)\mathbf{j}$ .

- a) Find the velocity vector.
- b) For what values of  $t$  in the interval  $[0, 2\pi]$  is  $\frac{d\mathbf{R}}{dt}$  equal to  $\mathbf{0}$ ?
- c) Find a Cartesian equation for a curve that contains the particle's path.

**1607.** At time  $t = 0$ , a particle is located at the point  $(1, 2)$ . It travels in a straight line to the point  $(4, 1)$ , has speed 2 at  $(1, 2)$ , and constant acceleration  $3\mathbf{i} - \mathbf{j}$ . Find an equation for the position vector  $\mathbf{R}(t)$  of the particle at time  $t$ .

**1608.** The position of a kite is given by  $\mathbf{R}(t) = \frac{1}{8}t\mathbf{i} - \frac{3}{64}t(t - 160)\mathbf{j}$ , where  $t \geq 0$  is measured in meters.

- a) How long is the kite above the ground?
- b) How high is the kite at  $t = 40$  seconds?
- c) At what rate is the kite's altitude increasing at  $t = 40$  seconds?
- d) At what time does the kite start to lose altitude?

**1609.** The paths of two particles for  $t \geq 0$  are given by

$$\mathbf{R}_1(t) = (t - 3)\mathbf{i} + (t - 3)^2\mathbf{j} \quad \text{and} \quad \mathbf{R}_2(t) = \left(\frac{3}{2}t - 4\right)\mathbf{i} + \left(\frac{3}{2}t - 2\right)\mathbf{j}.$$

- a) Determine the exact time(s) at which the particles collide.
- b) Find the direction of motion of each particle at the time(s) of collision.
- c) Find the speed of each particle at the time(s) of collision.

### 7.13 Polar Basics

IDENTIFY EACH OF THE FOLLOWING FUNCTIONS AS A *line*, *circle*, *cardioid*, *limaçon*, *lemniscate*, OR *rose*. IF THE EQUATION DESCRIBES A LIMAÇON, INDICATE WHETHER IT HAS AN INNER LOOP. IF THE EQUATION DESCRIBES A ROSE, INDICATE THE NUMBER OF PETALS.

1610.  $r = 3 - 2 \sin \theta$

1619.  $\theta = \frac{\pi}{3}$

1611.  $r = 6 \cos(3\theta)$

1620.  $r = 2\sqrt{\cos(2\theta)}$

1612.  $r = 1 + \cos \theta$

1621.  $r = \sqrt{6} \sin(3\theta)$

1613.  $r = \sqrt{9 \sin(2\theta)}$

1622.  $r = 5 \sin \theta$

1614.  $r = 7 \sec \theta$

1623.  $r = 3 \cos \theta$

1615.  $r = 7$

1624.  $r = 4$

1616.  $r = 2 - 5 \sin \theta$

1625.  $r \sin \theta = 9$

1617.  $r = 10 \cos \theta$

1626.  $\theta = \frac{\pi}{6}$

1618.  $r = 5 \sin(2\theta)$

1627.  $r^2 = 4 \sin(2\theta)$

FIND THE INTERSECTION POINTS OF THE FOLLOWING PAIRS OF POLAR CURVES.

1628.  $r = 2 + 2 \cos \theta$  and  $r = 3$

1629.  $r = 1 + \cos \theta$  and  $r = 1 - \cos \theta$

1630.  $r = 2 \sin \theta$  and  $r = 2 \sin(2\theta)$

1631.  $r = 1$  and  $r^2 = 2 \sin(2\theta)$

CONVERT THE RECTANGULAR EQUATION TO A POLAR EQUATION.

1632.  $x^2 + y^2 = 4$

1635.  $y^2 = 4x$

1633.  $y = 1$

1636.  $x = 1$

1634.  $xy = 2$

1637.  $x^2 + y^2 = e^{2 \arctan(y/x)}$





## 7.15 Sample A.P. Problems on Series, Vectors, Parametrics, and Polar

**1654.** A particle is moving along the path of the curve determined by the parametric equations  $x = e^{2t} + 1$  and  $y = \ln(e^{4t} + 2e^{2t} + 1)$ , where  $t > 0$ .

- Sketch the path of the curve and indicate the direction of motion.
- Find  $dy/dx$  in terms of  $t$ , then find the equation of the tangent line at time  $t = \frac{1}{2}$ .
- Show that  $\frac{d^2y}{dx^2} = \frac{-2}{(e^{2t} + 1)^2}$ .
- Write the set of parametric equations without the parameter  $t$ .

**1655.** Write the first three nonzero terms and the general term of the Taylor series generated by  $e^x$  at  $x = 0$ .

- Using the series above, write the first three nonzero terms and the general term of the Taylor series at  $x = 0$  for  $g(x) = \frac{e^x - 1}{x}$ .

- For the function  $g$  in part (a), find  $g'(1)$  and use it to show that  $\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1$ .

**1656.** Consider the family of polar curves defined by  $r = 2 + \cos(k\theta)$ , where  $k$  is a positive integer.

- Show that the area of the region enclosed by the curve does not depend on the value of  $k$ . What is the area?
- Write an expression in terms of  $k$  and  $\theta$  for the slope  $dy/dx$  of the curve.
- Find the value of  $dy/dx$  at  $\theta = \frac{\pi}{4}$  if  $k$  is a multiple of 4.

**1657 (1988BC).** Determine all values of  $x$  for which the series  $\sum_{k=0}^{\infty} \frac{2^k x^k}{\ln(k+2)}$  converges. Justify your answer.

**1658 (1992BC).** Consider the series  $\sum_{n=2}^{\infty} \frac{1}{n^p (\ln n)}$ , where  $p \geq 0$ .

- Show that the series converges for  $p > 1$ .
- Determine whether the series converges or diverges for  $p = 1$ . Show your analysis.
- Show that the series diverges for  $0 \leq p < 1$ .

- 1659 (1991BC).** Let  $f(t) = \frac{4}{1+t^2}$  and  $G(x) = \int_0^x f(t) dt$ .
- Find the first four nonzero terms and the general term for a power series for  $f(t)$  centered at  $t = 0$ .
  - Find the first four nonzero terms and the general term for a power series for  $G(x)$  centered at  $t = 0$ .
  - Find the interval of convergence for the power series in part (a). Justify your answer.
  - The interval of convergence of the power series in part (b) is almost the same as in part (c), but contains two more numbers. What are the numbers?
- 1660 (1998BC).** A particle moves along the curve defined by the equation  $y = x^3 - 3x$ . The  $x$ -coordinate of the particle,  $x(t)$ , satisfies the equation  $\frac{dx}{dt} = \frac{1}{\sqrt{2t+1}}$ , for  $t \geq 0$  with initial condition  $x(0) = -4$ .
- Find  $x(t)$  in terms of  $t$ .
  - Find  $dy/dt$  in terms of  $t$ .
  - Find the location and speed of the particle at time  $t = 4$ .
- 1661 (1990BC).** Let  $R$  be the region inside the graph of the polar curve  $r = 2$  and outside the graph of the polar curve  $r = 2(1 - \sin \theta)$ .
- Sketch the two polar curves and shade the region  $R$ .
  - Find the area of  $R$ .
- 1662 (1992BC).** At time  $t$ , for  $0 \leq t \leq 2\pi$ , the position of a particle moving along a path in the  $xy$ -plane is given by the parametric equations  $x = e^t \sin t$ ,  $y = e^t \cos t$ .
- Find the slope of the path of the particle at time  $t = \frac{\pi}{2}$ .
  - Find the speed of the particle when  $t = 1$ .
  - Find the distance traveled by the particle along the path from  $t = 0$  to  $t = 1$ .
- 1663 (1995BC).** Let  $f$  be a function that has derivatives of all orders for all real numbers. Assume  $f(1) = 3$ ,  $f'(1) = -2$ ,  $f''(1) = 2$  and  $f'''(1) = 4$ .
- Write the second-degree Taylor polynomial for  $f$  at  $x = 1$  and use it to approximate  $f(0.7)$ .
  - Write the third-degree Taylor polynomial for  $f$  at  $x = 1$  and use it to approximate  $f'(1.2)$ .
  - Write the second-degree Taylor polynomial for  $f'$  at  $x = 1$  and use it to approximate  $f'(1.2)$ .

**1664 (2001BC).** Let  $f(x) = \frac{1}{3} + \frac{2}{3^2}x + \frac{3}{3^3}x^2 + \cdots + \frac{n+1}{3^{n+1}}x^n + \cdots$  for all  $x$  in the interval of convergence.

- Find the interval of convergence. Show the work that leads to your conclusion.
- Find  $\lim_{x \rightarrow 0} \frac{f(x) - \frac{1}{3}}{x}$ .
- Write the first three nonzero terms and the general term for an infinite series that represents  $\int_0^1 f(x) dx$ .
- Find the sum of the series in part (c).

**1665 (1995BC).** Two particles move in the  $xy$ -plane. For time  $t \geq 0$ , the position of particle  $A$  is given by  $x = t - 2$  and  $y = (t - 2)^2$ , and the position of particle  $B$  is given by  $x = \frac{3}{2}t - 4$  and  $y = \frac{3}{2}t - 2$ .

- Find the velocity vector for each particle at time  $t = 3$ .
- Set up an integral expression that gives the distance traveled by particle  $A$  from  $t = 0$  to  $t = 3$ . Do not evaluate.
- Determine the exact time at which the particles collide; that is, when the particles are at the same point at the same time. Justify your answer.
- Sketch the paths of particles  $A$  and  $B$  from  $t = 0$  until they collide. Indicate the direction of each particle along its path.

**1666 (1997BC, Calculator).** During the time period from  $t = 0$  to  $t = 6$  seconds, a particle moves along the path given by  $x(t) = 3 \cos(\pi t)$  and  $y(t) = 5 \sin(\pi t)$ .

- Find the position of the particle when  $t = 2.5$ .
- Sketch the graph of the path of the particle from  $t = 0$  to  $t = 6$ . Indicate the direction of the particle along this path.
- How many times does the particle pass through the point found in part (a)?
- Find the velocity vector for the particle at any time  $t$ .
- Write and evaluate an integral expression, in terms of sine and cosine, that gives the distance the particle travels from time  $t = 1.25$  to  $t = 1.75$ .

## 7.16 Sample Multiple-Choice Problems on Series, Vectors, Parametrics, and Polar

**1667.** Which of the following series converge?

A)  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3^n}{n^2}$     B)  $\sum_{n=1}^{\infty} \frac{5n+7}{7n+5}$     C)  $\sum_{n=1}^{\infty} n e^{-n^2}$     D)  $\sum_{n=1}^{\infty} \frac{1}{(n^2+1)^{1/3}}$     E) None of these

**1668.** Which of the following sequences diverge?

A)  $\left\{ \frac{1}{n} \right\}$     B)  $\left\{ \frac{(-1)^{n+1}}{n} \right\}$     C)  $\left\{ \frac{2^n}{e^n} \right\}$     D)  $\left\{ \frac{n^2}{e^n} \right\}$     E)  $\left\{ \frac{n}{\ln n} \right\}$

**1669.** Which of the following series diverge?

A)  $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$     B)  $\sum_{n=1}^{\infty} \frac{n+1}{n!}$     C)  $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$     D)  $\sum_{n=1}^{\infty} \frac{\ln n}{2^n}$     E)  $\sum_{n=1}^{\infty} \frac{n}{2^n}$

**1670.** If  $\{a_n\} = \left\{ 1 + \frac{(-1)^n}{n} \right\}$  then

- A)  $\{a_n\}$  diverges by oscillation.
- B)  $\{a_n\}$  converges to zero.
- C)  $\{a_n\}$  converges to 1.
- D)  $\{a_n\}$  diverges to infinity.
- E)  $\{a_n\}$  converges to  $e^{-1}$ .

**1671.** The sum of the series  $2 - 1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \dots$  is

A)  $\frac{4}{3}$     B)  $\frac{5}{4}$     C) 1    D)  $\frac{3}{2}$     E)  $\frac{3}{4}$

**1672.** Which of the following is a term in the Taylor series about  $x = 0$  for the function  $f(x) = \cos 2x$ ?

A)  $-\frac{1}{2}x^2$     B)  $-\frac{4}{3}x^3$     C)  $\frac{2}{3}x^4$     D)  $\frac{1}{60}x^5$     E)  $\frac{4}{45}x^6$

**1673.** A curve in the  $xy$ -plane is defined by the parametric equations  $x = t^3 + 2$  and  $y = t^2 - 5t$ . Find the slope of the line tangent to the curve at the point where  $x = 10$ .

A)  $-12$     B)  $-\frac{3}{5}$     C)  $-\frac{1}{8}$     D)  $-\frac{1}{12}$     E)  $\frac{1}{20}$

**1674.** Find the values of  $x$  for which the series  $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n(-3)^n}$  converges.

A) 2 only    B)  $-1 \leq x < 5$     C)  $-1 < x \leq 5$     D)  $-1 < x < 5$     E) all real numbers

**1675.** The area enclosed by the graph of  $r = 5 \cos(4\theta)$  is

A) 5    B) 10    C)  $\frac{25\pi}{4}$     D)  $\frac{25\pi}{2}$     E)  $25\pi$

**1676.** The position vector of a particle moving in the  $xy$ -plane is given by  $\mathbf{r}(t) = \langle \sin^{-1} t, (t + 4)^2 \rangle$  for  $-1 \leq t \leq 1$ . The velocity vector at  $t = 0.6$  is

- A)  $\langle \sin^{-1}(0.6), (4.6)^2 \rangle$     B)  $\langle \frac{5}{4}, 9.2 \rangle$     C)  $\langle \frac{5}{3}, 1.2 \rangle$     D)  $\langle \frac{5}{3}, 9.2 \rangle$     E)  $\langle \frac{75}{64}, 2 \rangle$

**1677.** Let  $\sum_{n=1}^{\infty} u_n$  be a series for which  $\lim_{n \rightarrow \infty} u_n = 0$ . Which of the following statements is always true?

- A)  $\sum_{n=1}^{\infty} u_n$  converges to a finite sum.  
B)  $\sum_{n=1}^{\infty} u_n = 0$ .  
C)  $\sum_{n=1}^{\infty} u_n$  does not diverge to infinity.  
D)  $\sum_{n=1}^{\infty} u_n$  is a positive series.  
E) None of the above are always true.

**1678.** The Taylor polynomial of order 3 at  $x = 0$  for  $f(x) = \sqrt{1+x}$  is

- A)  $1 + \frac{1}{2}x - \frac{1}{4}x^2 + \frac{3}{8}x^3$   
B)  $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$   
C)  $1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{16}x^3$   
D)  $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{8}x^3$   
E)  $1 - \frac{1}{2}x + \frac{1}{4}x^2 - \frac{3}{8}x^3$

**1679.** The series  $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots$  converges if and only if

- A)  $-1 < x < 1$  only    B)  $-1 \leq x \leq 1$     C)  $-1 \leq x < 1$     D)  $-1 < x \leq 5$     E)  $x = 0$

**A.P. Calculus BC Test Six**  
**Section One**  
**Multiple-Choice**  
**No Calculators**  
Time—35 minutes  
Number of Questions—15

The scoring for this section is determined by the formula

$$[C - (0.25 \times I)] \times 1.8$$

where  $C$  is the number of correct responses and  $I$  is the number of incorrect responses. An unanswered question earns zero points. The maximum possible points earned on this section is 27, which represents 50% of the total test score.

*Directions:* Solve each of the following problems, using the available space for scratch work. After examining the form of the choices, decide which is the best of the choices given and fill in the corresponding choice on your answer sheet. Do not spend too much time on any one problem.

Good Luck!

NAME:

1. Which of the following series is convergent?

A)  $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$

B)  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

C)  $\sum_{n=1}^{\infty} \frac{1}{n}$

D)  $\sum_{n=1}^{\infty} \frac{1}{10n - 1}$

E)  $\sum_{n=1}^{\infty} \frac{2}{n^2 - 5}$

---

2. Which of the following series is divergent?

A)  $\sum_{n=1}^{\infty} \frac{1}{n^2}$

B)  $\sum_{n=1}^{\infty} \frac{1}{n^2 + n}$

C)  $\sum_{n=1}^{\infty} \frac{n}{n^3 + 1}$

D)  $\sum_{n=1}^{\infty} \frac{n}{\sqrt{4n^2 - 1}}$

E) None of these

---

3. The position of a particle moving from the origin in the  $xy$ -plane at any time  $t$  is given by the vector  $\mathbf{r} = (3 \cos \frac{\pi t}{3}) \mathbf{i} + (2 \sin \frac{2\pi}{3}) \mathbf{j}$ . The magnitude of the acceleration when  $t = 3$  is

A) 2

B)  $\frac{\pi^2}{3}$

C) 3

D)  $\frac{2\pi^2}{9}$

E)  $\pi$



4. The series  $(x - 2) + \frac{(x - 2)^2}{4} + \frac{(x - 2)^3}{9} + \frac{(x - 2)^4}{16} + \dots$  converges for

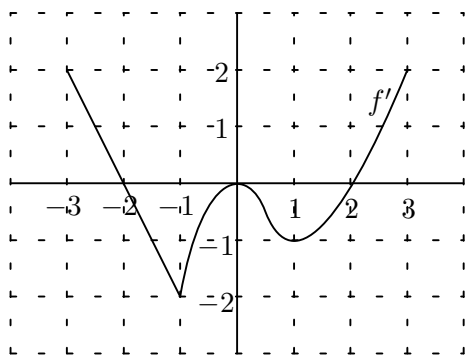
- A)  $1 \leq x \leq 3$
  - B)  $1 \leq x < 3$
  - C)  $1 < x \leq 3$
  - D)  $0 \leq x \leq 4$
  - E) None of these
- 

5. Which of the following statements about series is false?

- A)  $\sum_{n=1}^{\infty} a_n = \sum_{n=k}^{\infty} a_n$  where  $k$  is any positive integer.
  - B) If  $\sum_{n=1}^{\infty} a_n$  converges, then so does  $\sum_{n=1}^{\infty} ca_n$  where  $c \neq 0$ .
  - C) If  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  converge, then so does  $\sum_{n=1}^{\infty} (ca_n + b_n)$  where  $c \neq 0$ .
  - D) If 1000 terms are added to a convergent series, the new series also converges.
  - E) Rearranging the terms of a positive convergent series will not affect its convergence or its sum.
- 

6. Find the area inside the polar curve  $r = 3 \cos 3\theta$ .

- A)  $\frac{7\pi}{4}$
- B)  $2\pi$
- C)  $\frac{9\pi}{4}$
- D)  $\frac{5\pi}{2}$
- E)  $\frac{11\pi}{4}$



7. Above is the graph of  $f'(x)$ , the derivative of  $f(x)$ . The domain of  $f$  is the interval  $-3 \leq x \leq 3$ . Which of the following are true about the graph of  $f$ ?

- I.  $f$  is increasing on  $-3 < x < -2$ .
- II.  $f$  is concave down on  $-3 < x < -1$ .
- III. The maximum value of  $f(x)$  on  $-3 < x < 2$  is  $f(-3)$ .

- A) I only
- B) II only
- C) III only
- D) I and II only
- E) II and III only

8. The sales of a small company are expected to grow at a rate given by  $\frac{dS}{dt} = 300t + t^{1/2} + t^{3/2}$ , where  $S(t)$  is the sales in dollars in  $t$  days. The accumulated sales through the first 4 days is approximately

- A) \$2202
- B) \$2274
- C) \$2346
- D) \$2418
- E) \$2490

9. The radius of convergence of the series  $\frac{x}{4} + \frac{x^2}{4^2} + \frac{x^3}{4^3} + \cdots + \frac{x^n}{4^n} + \cdots$  is

- A)  $\infty$
- B) 0
- C) 1
- D) 2
- E) 4

10. The position vector of a particle moving in the  $xy$ -plane at time  $t$  is given by

$$\mathbf{p} = (3t^2 - 4t)\mathbf{i} + (t^2 + 2t)\mathbf{j}.$$

The speed of the particle at  $t = 2$  is

- A) 2 units per second.
  - B)  $2\sqrt{10}$  units per second.
  - C) 10 units per second.
  - D) 14 units per second.
  - E) 20 units per second.
- 

11. The coefficient of  $x^3$  in the Taylor series for  $e^{2x}$  at  $x = 0$  is

- A)  $\frac{1}{6}$ .
  - B)  $\frac{1}{3}$ .
  - C)  $\frac{2}{3}$ .
  - D)  $\frac{4}{3}$ .
  - E)  $\frac{8}{3}$ .
- 

12. Which of the following is an equation for the line tangent to the curve with parametric equations

$$x = \frac{1}{t} \quad \text{and} \quad y = \sqrt{t+1}$$

at the point where  $t = 3$  ?

- A)  $-\frac{4}{9}\left(x - \frac{1}{3}\right) = y - 2$
- B)  $\frac{1}{4}\left(x - \frac{1}{3}\right) = y - 2$
- C)  $-\frac{9}{4}\left(x - \frac{1}{3}\right) = y - 2$
- D)  $-\frac{4}{9}\left(x + \frac{1}{9}\right) = y - \frac{1}{4}$
- E)  $-\frac{9}{4}\left(x + \frac{1}{9}\right) = y - \frac{1}{4}$

13. The area inside the circle with polar equation  $r = 2 \sin \theta$  and above the lines with equations  $y = x$  and  $y = -x$  is given by

A)  $\int_{-\pi/4}^{\pi/4} 2 \sin^2 \theta \, d\theta$

B)  $\int_{-1}^1 2 \sin \theta \, d\theta$

C)  $\int_{-1}^1 (2 \sin^2 \theta - 1) \, d\theta$

D)  $\int_{\pi/4}^{3\pi/4} \sin \theta \, d\theta$

E)  $\int_{\pi/4}^{3\pi/4} 2 \sin^2 \theta \, d\theta$

---

14. What is the sum  $\frac{5}{2} + \frac{5}{4} + \frac{5}{8} + \frac{5}{16} + \dots$  ?

A) 2

B)  $\frac{75}{16}$

C)  $\frac{315}{64}$

D) 5

E) This series diverges

---

15. Suppose  $f$  is a function whose  $n$ th derivative is  $f^{(n)}(x) = (2^x + 1)(n + 1)!$  for all  $x$  and  $n$ . If  $f(3) = -2$ , what is the fourth-degree Taylor polynomial for  $f$  at  $x = 3$  ?

A)  $-2 + 18(x - 3) + 27(x - 3)^2 + 36(x - 3)^3 + 45(x - 3)^4$

B)  $-2 + 18x + 27x^2 + 36x^3 + 45x^4$

C)  $-2 + 18(x - 3) + 54(x - 3)^2 + 216(x - 3)^3 + 1080(x - 3)^4$

D)  $-2 + 18x + 54x^2 + 216x^3 + 1080x^4$

E)  $-2 + 18(x - 3) + 27(x - 3)^2 + 72(x - 3)^3 + 270(x - 3)^4$

**A.P. Calculus BC Test Six**  
**Section Two**  
**Free-Response**  
**Calculators Allowed**  
Time—45 minutes  
Number of Questions—3

Each of the three questions is worth 9 points. The maximum possible points earned on this section is 27, which represents 50% of the total test score. There is no penalty for guessing.

- **SHOW ALL YOUR WORK.** You will be graded on the methods you use as well as the accuracy of your answers. Correct answers without supporting work may not receive full credit.
- Write all work for each problem in the space provided. Be sure to write clearly and legibly. Erased or crossed out work will not be graded.
- Justifications require that you give mathematical (non-calculator) reasons and that you clearly identify functions, graphs, tables, or other objects that you use.
- You are permitted to use your calculator to solve an equation or graph a function without showing work. However, you must clearly indicate the setup of your problem.
- Your work must be expressed in mathematical notation rather than calculator syntax. For example,  $\int_1^5 x^2 dx$  may not be written as `fnInt(X^2,X,1,5)`.
- Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Good Luck!

NAME:

1. A particle moves in the  $xy$ -plane in such a manner that its coordinates at time  $t$  are  $x = 3 \cos \frac{\pi}{4}t$  and  $y = 5 \sin \frac{\pi}{4}t$ .

- a) Find the length of the velocity vector at  $t = 3$ .
  - b) Find the  $x$ - and  $y$ -coordinates of the acceleration of the particle at  $t = 3$ .
  - c) Find a single equation in  $x$  and  $y$  for the path of the particle.
- 

2. Consider the polar curve  $r = 2 \sin 3\theta$  for  $0 \leq \theta \leq \pi$ .

- a) Sketch the curve.
  - b) Find the area of the region inside the curve.
  - c) Find an expression, in terms of  $\theta$ , for the slope  $dy/dx$  of the curve, then find the slope when  $\theta = \pi/4$ .
- 

3. Let  $f$  be a function that has derivatives of all orders for all real numbers. Assume  $f(0) = 5$ ,  $f'(0) = -3$ ,  $f''(0) = 1$ , and  $f'''(0) = 4$ .

- a) Write the third-degree Taylor polynomial for  $f$  at  $x = 0$  and use it to approximate  $f(0.2)$ .
- b) Write the fourth-degree Taylor polynomial for  $g$ , where  $g(x) = f(x^2)$ , at  $x = 0$ .
- c) Write the third-degree Taylor polynomial for  $h$ , where  $h(x) = \int_0^x f(t) dt$ , at  $x = 0$ .
- d) Let  $h$  be defined as in part (c). Given that  $f(1) = 3$ , either find the exact value of  $h(1)$  or explain why it cannot be determined.

## CHAPTER 8

---

# AFTER THE A.P. EXAM

---

## 8.1 Hyperbolic Functions

$$\begin{array}{ll} \cosh x = \frac{e^x + e^{-x}}{2} & \sinh x = \frac{e^x - e^{-x}}{2} \\ \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} & \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} \\ \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} & \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} \end{array}$$

**1680.** Sketch the graphs of all six of the above functions.

**1681.** Establish the following identities.

a)  $\cosh^2 x - \sinh^2 x = 1$

d)  $\sinh 2x = 2 \sinh x \cosh x$

b)  $1 - \tanh^2 x = \operatorname{sech}^2 x$

e)  $\cosh 2x = \cosh^2 x + \sinh^2 x$

c)  $\coth^2 x - 1 = \operatorname{csch}^2 x$

f)  $\sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$

**1682.** Find the derivatives of all six hyperbolic functions.

FIND THE FOLLOWING DERIVATIVES.

**1683.**  $\ln(\cosh x)$

**1686.**  $\tanh^2 x$

**1684.**  $\frac{1}{2} \sinh(2x - 1)$

**1687.**  $\operatorname{sech}(\ln x)$

**1685.**  $\operatorname{sech} x [1 - \ln(\operatorname{sech} x)]$

**1688.**  $x^2 \tanh\left(\frac{1}{x}\right)$

EVALUATE THE FOLLOWING.

**1689.**  $\int \sinh 2x \, dx$

**1692.**  $\int_0^{\ln 2} 4e^{-x} \sinh x \, dx$

**1690.**  $\int 4 \cosh(3x - \ln 2) \, dx$

**1693.**  $\int_{-1}^1 2 \cosh x \, dx$

**1691.**  $\int \tanh x \, dx$

**1694.**  $\int_{\ln 2}^{\ln 4} \coth x \, dx$

**1695.** A cable suspended from its two ends hangs in the shape of a catenary, which is the graph of an equation of the form  $y = a \cosh\left(\frac{x}{a}\right)$ .

a) Calculate the length of a cable suspended from two poles 100 meters apart; i.e., for  $-50 \leq x \leq 50$ . Your answer will be in terms of  $a$ . Then evaluate the actual length of the cable for the following values of  $a$ : 50, 100, 200.

b) Note that the point  $(0, a)$  lies on the graph of the catenary and so  $a \cosh\left(\frac{50}{a}\right) - a$  measures the amount by which the cable “sags.” How does the amount of “sag” compare with the length of the cable? Does this strike you as paradoxical?



## 8.2 Surface Area of a Solid of Revolution

FOR THE FOLLOWING THREE PROBLEMS, FIND THE AREA OF THE SURFACE OF THE SOLID OBTAINED BY REVOLVING EACH CURVE OVER THE GIVEN INTERVAL ABOUT THE  $x$ -AXIS.

**1696.**  $y = 5x - 1$  over  $[0, 2]$

**1697.**  $y = x^3$  over  $[1, 2]$

**1698.**  $y = \sqrt{x}$  over  $[1, 4]$

FOR THE FOLLOWING FOUR PROBLEMS, SET UP THE INTEGRAL THAT REPRESENTS THE SURFACE AREA OF THE SOLID OBTAINED BY REVOLVING EACH CURVE OVER THE GIVEN INTERVAL ABOUT THE  $x$ -AXIS, THEN EVALUATE THE INTEGRAL USING YOUR CALCULATOR.

**1699.**  $y = x^2$  over  $[1, 4]$

**1700.**  $y = e^x$  over  $[0, 1]$

**1701.**  $y = \sin x$  over  $[0, \pi]$

**1702.**  $y = x^3 - 3x^2 - 6x + 19$  over  $[1, 10]$

**1703.** Use an integral to find the surface area of a sphere of radius 3.

**1704.** Find the surface area of a solid generated by revolving the region bounded by  $y = 3\sqrt{x}$ ,  $x = 9$ ,  $x = 16$ , and the  $x$ -axis about the  $x$ -axis.

**1705.** Consider the region in the first quadrant bounded by  $y = \frac{1}{x}$ ,  $x = 1$ , and  $x = b$  for some  $b > 1$ .

- Find  $V(b)$ , the volume of the solid generated by revolving the region about the  $x$ -axis.
- Find  $S(b)$ , the surface area of the solid generated by revolving the region about the  $x$ -axis.
- To what values do  $V(b)$  and  $S(b)$  approach as  $b \rightarrow \infty$ ?

**1706.** An ornamental light bulb is designed by revolving the graph of  $y = \frac{1}{3}x^{1/2} - x^{3/2}$  for  $0 \leq x \leq \frac{1}{3}$  about the  $x$ -axis. Both  $x$  and  $y$  are measured in feet, and the glass used to make the bulb is 0.015 inches thick. Find the surface area of the bulb and use the result to approximate the amount of glass needed to make the bulb.

---

A modern branch of mathematics, having achieved the art of dealing with the infinitely small, can now yield solutions in other more complex problems of motion, which used to appear insoluble. This modern branch of mathematics, unknown to the ancients, when dealing with problems of motion, admits the conception of the infinitely small, and so conforms to the chief condition of motion (absolute continuity) and thereby corrects the inevitable error which the human mind cannot avoid when dealing with separate elements of motion instead of examining continuous motion. In seeking the laws of historical movement just the same thing happens. The movement of humanity, arising as it does from innumerable human wills, is continuous. To understand the laws of this continuous movement is the aim of history. Only by taking an infinitesimally small unit for observation (the differential of history, that is, the individual tendencies of man) and attaining to the art of integrating them (that is, finding the sum of these infinitesimals) can we hope to arrive at the laws of history. —*Leo Tolstoy*

### 8.3 Linear First Order Differential Equations

The equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

has the solution

$$y = \frac{1}{v(x)} \int v(x)Q(x) dx$$

where  $v(x) = e^{\int P(x) dx}$ .

SOLVE THE FOLLOWING DIFFERENTIAL EQUATIONS.

**1707.**  $\frac{dy}{dx} + 4y = e^{-2x}$

**1710.**  $x \frac{dy}{dx} + 3y = \frac{\sin x}{x^2}$

**1713.**  $(\tan x) \frac{dy}{dx} + y = \sin^2 x$

**1708.**  $\frac{dy}{dx} - 4y = e^{3x}$

**1711.**  $(1+x) \frac{dy}{dx} + y = \sqrt{x}$

**1714.**  $\frac{dy}{dx} - y = x$

**1709.**  $\frac{dy}{dx} + 2y = 2 \cosh x$

**1712.**  $x \frac{dy}{dx} - y = 2x \ln x$

SOLVE THE INITIAL-VALUE PROBLEMS.

**1715.**  $\frac{dy}{dx} + 2y = 3, \quad y(0) = 1$

**1717.**  $\frac{dy}{dx} + xy = x, \quad y(0) = -6$

**1716.**  $x \frac{dy}{dx} + y = \sin x, \quad x > 0, \quad y\left(\frac{\pi}{2}\right) = 1$

**1718.**  $\frac{dy}{dx}(\cos x) - y \sin x = \sin(2x), \quad y(0) = 1$

**1719.** A chemical in a liquid solution runs into a tank holding the liquid with a certain amount of the chemical already dissolved. The mixture is kept uniform by stirring and runs out of the tank at a known rate. In this process, we want to know the concentration of the chemical in the tank at any given time. If  $C(t)$  is the amount of chemical in the tank at time  $t$  and  $V(t)$  is the volume of liquid in the tank at time  $t$ , then the rate of change in the amount of the chemical is given by

$$\frac{dC}{dt} = (\text{chemical's inflow rate}) - \frac{C(t) \cdot \text{outflow rate}}{V(t)}.$$

Consider the following. A tank initially contains 100 gal of brine in which 50 pounds of salt are dissolved. A brine containing 2 pounds per gal of salt runs into the tank at the rate of 5 gal/min. The mixture is kept uniform by stirring and flows out of the tank at a rate of 4 gal/min.

- At what rate does salt enter the tank at time  $t$ ?
- What is the volume of brine in the tank at time  $t$ ?
- At what rate does salt leave the tank at time  $t$ ?
- Write down and solve the initial value problem describing the mixing process.
- Find the concentration of salt in the tank 25 minutes after the process starts.

## 8.4 Curvature

Isaac Newton discovered a method for comparing the degree of curvature of a function with the curvature of a circle. The curvature of the graph of some function  $y$  of  $x$  is defined to be

$$\kappa = \frac{|y''|}{[1 + (y')^2]^{3/2}}.$$

( $\kappa$  is the Greek letter kappa.) To determine the curvature at a certain point  $x = c$ , you simply evaluate both first and second derivatives at  $x = c$  then compute  $\kappa$ . The quantity  $\kappa$  tells us that the function  $y$  at the point  $c$  has the same degree of curvature as a circle of radius 1.

For example, to find the curvature of  $y = x^2$ , compute the derivatives  $y' = 2x$  and  $y'' = 2$ . Then

$$\kappa = \frac{|2|}{[1 + (2x)^2]^{3/2}} = \frac{2}{[1 + 4x^2]^{3/2}}.$$

At the particular point  $x = \frac{1}{2}$ ,

$$\kappa = \frac{2}{[2]^{3/2}} = \frac{1}{\sqrt{2}}.$$

Thus  $x^2$  at the point  $x = \frac{1}{2}$  has the same curvature as a circle of radius  $\frac{1}{\kappa} = \sqrt{2}$ .

**1720.** Find the curvature of  $y = mx + b$  where  $m$  and  $b$  are real numbers. Does this result make sense? Why or why not?

**1721.** Find the curvature of  $y = x^3$  at  $x = 0$ .

**1722.** Find the curvature of  $y = \cosh x$  at  $x = 0$ .

**1723.** Find the curvature of  $y = e^x$  at  $x = 0$ . What is the radius of the circle with the same curvature? Give both exact and approximate answers.

**1724.** Find the curvature of  $y = \ln x$  at  $x = 1$ . What is the radius of the circle with the same curvature? Explain how this answer compares with the answer to the previous problem.

**1725.** Clearly,  $\kappa = 0$  only if  $y''(c) = 0$ . If  $y$  is any non-linear function, at what *type of point*  $c$  will the curvature always be zero?

**1726.** Which is more “curvy” at  $x = \frac{\pi}{4}$ :  $y = \cos x$  or  $y = \ln(\cos x)$ ? Justify your answer.

---

To arrive at the simplest truth, as Newton knew and practiced, requires years of contemplation. Not activity. Not reasoning. Not calculating. Not busy behaviour of any kind. Not reading. Not talking. Not making an effort. Not thinking. Simply bearing in mind what it is one needs to know. And yet those with the courage to tread this path to real discovery are not only offered practically no guidance on how to do so, they are actively discouraged and have to set about it in secret, pretending meanwhile to be diligently engaged in the frantic diversions and to conform with the deadening personal opinions which are continually being thrust upon them. —George Spencer Brown

## 8.5 Newton's Method

COMPLETE ONE ITERATION OF NEWTON'S METHOD FOR THE GIVEN FUNCTION USING THE INDICATED INITIAL GUESS.

**1727.**  $f(x) = x^2 - 3$ ;  $x_0 = 1.7$

**1728.**  $f(x) = 3x^2 - 23$ ;  $x_0 = 1$

APPLY NEWTON'S METHOD USING THE INDICATED INITIAL GUESS AND EXPLAIN WHY THE METHOD FAILS.

**1729.**  $f(x) = 2x^3 - 6x^2 + 6x - 1$ ;  $x_0 = 1$

**1730.**  $f(x) = 4x^3 - 12x^2 + 12x - 3$ ;  $x_0 = 1.5$

USE NEWTON'S METHOD TO ESTIMATE THE POINT OF INTERSECTION OF THE GIVEN GRAPHS USING THE INITIAL GUESS.

**1731.**  $y = \tan x$  and  $y = 2x$ ;  $x_0 = 1.25$

**1732.**  $y = x^2$  and  $y = \cos x$ ;  $x_0 = 1$

**1733.**  $x^2 - 2 = 0$  has two roots,  $\sqrt{2}$  and  $-\sqrt{2}$ . Which values of  $x_0$  lead to which root? Draw a graph to explain.

**1734.** If the initial approximation of a root of  $x^2 - 5 = 0$  is given as  $x_0 = 2$  then the next approximation produced by Newton's Method would be what?

**1735.** The function  $f$  defined by  $f(x) = x^3 - 3x + 1$  is a cubic polynomial with three real roots, and yet an incautious choice of  $x_0$  can lead to surprising results.

a) For each of the following values of  $x_0$ , find out to which root (if any) Newton's Method converges.

i) 1.05

ii) 1

iii) 0.95

iv) 0.911

v) 0.91

vi) 0.8893406567

vii) 0.85

b) Based on this experience, what precautions might you take in choosing  $x_0$ ?

---

Each generation has its few great mathematicians, and mathematics would not even notice the absence of the others. They are useful as teachers, and their research harms no one, but it is of no importance at all. A mathematician is great or he is nothing. —*Alfred Adler*

## CHAPTER 9

---

# **PRACTICE and REVIEW**

---

## 9.1 Algebra

*For use anytime after Section 1.15*

SOLVE EACH OF THE FOLLOWING FOR  $y$ .

**1736.**  $x^2 + y^2 = 9$

**1739.**  $\ln y = x^2 + 5$

**1737.**  $\frac{x^2}{9} - \frac{y^2}{4} = 1$

**1740.**  $e^y = x + 7$

**1738.**  $\frac{(x-1)^2}{8} + \frac{(y+1)^2}{4} = 1$

**1741.**  $xy - y + x - 1 = 1$

FIND ALL REAL ZEROS OF THE FOLLOWING FUNCTIONS.

**1742.**  $y = 2x(x-1)^2$

**1743.**  $y = x^3 + 5x^2 + 4x + 20$

**1744.**  $y = x^3(x+2)$

**1745.**  $y = \ln(x-5)\sqrt{x^3-8}$

FOR EACH PAIR OF FUNCTIONS, FIND  $f(g(x))$  AND ITS DOMAIN, THEN FIND  $g(f(x))$  AND ITS DOMAIN.

**1746.**  $f(x) = |x|$ ;  $g(x) = x^2 - 1$

**1747.**  $f(x) = 3x - 4$ ;  $g(x) = \frac{x+4}{3}$

**1748.**  $f(x) = \frac{1}{x^2} + 1$ ;  $g(x) = \frac{1}{x-1}$

**1749.**  $f(x) = x^2 + 1$ ;  $g(x) = \sqrt{x}$

THE TABLE BELOW GIVES VALUES OF TWO FUNCTIONS  $f$  AND  $g$ . USING THE TABLE, DETERMINE THE VALUES OF THE FOLLOWING COMPOSITIONS.

$x$		-4		-3		-2		-1		0		1		2
$f(x)$		$-\frac{1}{2}$		$-\frac{2}{3}$		-1		-2		undefined		2		1
$g(x)$		0		-2		0		2		2		0		-2

**1750.**  $f(g(-1))$

**1754.**  $f(g(-3))$

**1751.**  $g(f(2))$

**1755.**  $f(f(1))$

**1752.**  $f(g(0))$

**1756.**  $g(g(-4))$

**1753.**  $g(f(-1))$

**1757.**  $g(g(0))$

## 9.2 Derivative Skills

*For use anytime after Section 2.15*

FIND THE DERIVATIVE OF EACH FUNCTION IN SIMPLEST FACTORED FORM.

**1758.**  $T(x) = e^x(2x^2 + 3)$

**1759.**  $h(x) = (5^x)(2^{x-1})$

**1760.**  $o(x) = \ln \left[ \frac{(x+1)^{16}(2x^2+x)^8}{\sqrt{x^2+4}} \right]$

**1761.**  $m(x) = \ln \left[ \frac{(e^{2x}+6)^7 \sqrt{x+4}}{(e^{-x}+e^x)^5} \right]$

**1762.**  $a(x) = \ln \sqrt{\frac{2-3x}{x+4}}$

**1763.**  $s(x) = \frac{\log x}{3x}$

**1764.**  $P(x) = (4x-1) \sec 3x$

**1765.**  $o(x) = \frac{\tan 3x}{\tan 2x}$

**1766.**  $l(x) = \frac{\cos 2x}{\log x}$

**1767.**  $s(x) = x(1 + \cos^2 x)$

**1768.**  $t(x) = 3^{x^2}$

**1769.**  $r(x) = \log \left( \frac{x+1}{x^2+1} \right)$

**1770.**  $a(x) = 2xe^{-x}$

FIND THE DERIVATIVE IMPLICITLY.

**1771.**  $2x \ln y - 3y \ln x = 6xy$

**1772.**  $e^{3y} - 2 = \ln(x^2 - 4y)$

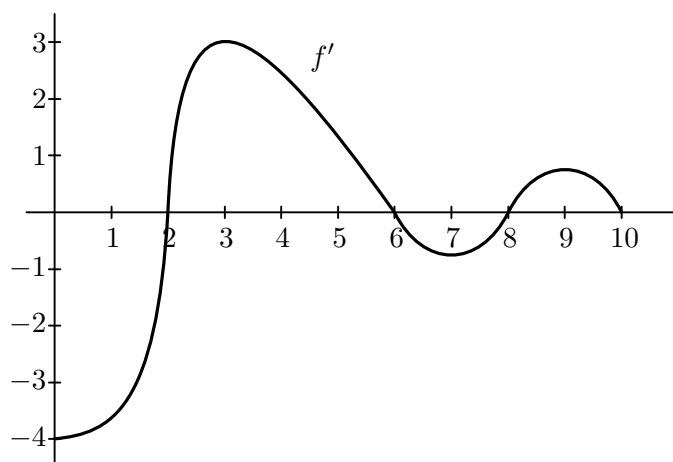
**1773.**  $xe^{5y} - yx^2 = \ln x$

---

All economical and practical wisdom is an extension or variation of the following arithmetical formula:  $2 + 2 = 4$ . Every philosophical proposition has the more general character of the expression  $a + b = c$ . We are mere operatives, empirics, and egotists, until we learn to think in letters instead of figures. —*Oliver Wendell Holmes*

### 9.3 Can You Stand All These Exciting Derivatives?

For use anytime after Section 3.8



**1774.** The graph above is the graph of the derivative of a function  $f$ . Use the graph to answer the following questions about  $f$  on the interval  $(0, 10)$ . Justify your answers.

- On what subinterval(s) is  $f$  increasing?
- On what subinterval(s) is  $f$  decreasing?
- Find the  $x$ -coordinates of all relative minima of  $f$ .
- Find the  $x$ -coordinates of all relative maxima of  $f$ .
- On what subinterval(s) is  $f$  concave up?
- On what subinterval(s) is  $f$  concave down?
- Find the  $x$ -coordinates of all points of inflection of  $f$ .

**1775.**  $F$ ,  $F'$ ,  $G$ , and  $G'$  have values as listed in the table below. Let  $P(x) = F(G(x))$ ,  $K(x) = F(x)/G(x)$ ,  $T(x) = F(x)G(x)$ ,  $R(x) = F(x) + G(x)$ , and  $N(x) = (G(x))^3$ . Use the table to evaluate the derivatives.

$x$	$F(x)$	$F'(x)$	$G(x)$	$G'(x)$
1	2	-4	-3	6
-3	1	5	-2	-1

- a)  $P'(1)$       b)  $K'(-3)$       c)  $T'(1)$       d)  $R'(-3)$       e)  $N'(-3)$

**1776.** The line tangent to the graph of a function  $g(x)$  at the point  $(-5, 4)$  passes through the point  $(0, -1)$ . Find  $g'(-5)$ .



- 1777.** Let  $y = -3x + 2$  be the tangent line to  $F(x)$  at  $x = 1$ . Find  $F(1)$  and  $F'(1)$ .
- 1778.** The line tangent to the graph of  $H$  at  $x = 3$  has slope  $-4$  and has an  $x$ -intercept at  $x = 6$ . Find  $H(3)$  and  $H'(3)$ .
- 1779 (Calculator).** Let  $F'(x) = 3 \cos(x) - e^x$ .
- Graph  $F'(x)$  in the window  $-5 < x < 2$ ,  $-10 < y < 10$ .
  - On what intervals is  $F(x)$  increasing on  $[-5, 2]$ ?
  - Where is  $F(x)$  concave down on  $[-5, 2]$ ?
  - Where does  $F(x)$  have a relative maximum on  $[-5, 2]$ ?
  - Find all inflection points of  $F(x)$  on  $[-5, 2]$ .
  - Sketch a possible graph of  $F(x)$  on  $[-5, 2]$ .
- 1780.** A particle is moving along a line with position function  $s(t) = 3 + 4t - 3t^2 - t^3$ .
- Find the velocity function.
  - Find the acceleration function.
  - Describe the motion of the particle for  $t \geq 0$ .
- 1781.** The positions of two particles on the  $x$ -axis are  $x_1 = \cos t$  and  $x_2 = \cos(t + \frac{\pi}{4})$ . What is the farthest apart the particles ever get? When do the particles collide?
- 1782.** The circumference of the equator of a sphere is measured as 10 cm with a possible error of 0.4 cm. This measurement is then used to calculate the radius. The radius is then used to calculate the surface area and volume of the sphere. Estimate the percentage error in the calculated values of a) the radius, b) the surface area, and c) the volume.
- 1783.** Graph the function  $a(x) = |x - 1| + |x - 3|$ .
- What is the domain of  $a(x)$ ? Is  $a(x)$  even, odd, or neither?
  - Find an expression for  $a'(x)$ . *Hint:*  $a'(x)$  is piecewise.
  - Find all relative extrema of  $a(x)$ .
- 1784.** For groups of 80 or more, a charter bus company determines the rate per person according to the following formula:
- $$\text{Rate} = \$8.00 - \$0.05(n - 80)$$
- for  $n \geq 80$ . What number of passengers will give the bus company maximum revenue?

## 9.4 Different Differentiation Problems

For use anytime after Section 3.8

**1785.** The temperature of an object at time  $t$  is given by  $T(t) = 0.1(t^4 - 12t^3 + 2000)$  for  $0 \leq t \leq 10$ .

- Find the hottest and coldest temperature during the interval  $[0, 10]$ .
- At what time is the rate of change in the temperature a minimum?

**1786.** Consider the two ellipses given by

$$x^2 + xy + y^2 = 1 \quad \text{and} \quad x^2 - xy + y^2 = 1.$$

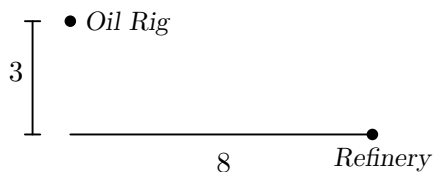
The first ellipse passes through the points  $(1, -1)$  and  $(-1, 1)$ ; the second passes through  $(1, 1)$  and  $(-1, -1)$ . The ellipses intersect in four points:  $(0, 1)$ ,  $(1, 0)$ ,  $(0, -1)$ , and  $(-1, 0)$ .

- Graph the ellipses.
- Find  $dy/dx$  for each ellipse.
- Find the slope at each intersection point.
- Find all points on each ellipse where the tangent line is horizontal.
- Find all points on each ellipse where the tangent line is vertical.
- Find the two lines that intersect both ellipses at right angles.

**1787.** Jay is a waiter at a fine-dining restaurant with 100 tables. During his first month he waited on 20 tables every night, and collected an average tip of \$15 from each table. He started to work more tables, and noticed that for every extra table he took on in a night, his average tip would go down 25 cents per table. He figures that he is physically capable of waiting on up to 30 tables in a night. If Jay wants to maximize his tip money, how many more tables should he wait on?

**1788.** A truck traveling on a flat interstate highway at a constant rate of 50 mph gets 8 miles to the gallon. Fuel costs \$2.30 per gallon. For each mile per hour increase in speed, the truck loses a fifth of a mile per gallon in its mileage. Drivers get \$27.50 per hour in wages, and fixed costs for running the truck amount to \$12.33 per hour. What constant speed should a dispatcher require on a straight run through 260 miles of Kansas interstate to minimize the total cost of operating the truck?

**1789.** Oil from an offshore rig located 3 miles from the shore is to be pumped to a refinery location on the edge of the shore that is 8 miles east of the rig. The cost of constructing a pipe along the ocean floor from the rig to shore is 1.5 times as expensive as the cost of constructing the pipe on land. How far to the west of the refinery should the pipe come on to shore in order to minimize cost?



**1790.** Let  $f(x) = \arctan x$ .

- Find  $f'(x)$ .
- Evaluate  $\lim_{x \rightarrow \infty} f(x)$ ,  $\lim_{x \rightarrow -\infty} f(x)$ ,  $\lim_{x \rightarrow \infty} f'(x)$ , and  $\lim_{x \rightarrow -\infty} f'(x)$ .
- Is  $f$  even, odd, or neither?
- Show that  $f$  is increasing over all real numbers.
- Where is  $f$  concave up? Concave down? Where are the inflection points, if any exist, of  $f$ ?
- How do the graphs of  $f$  and  $f'$  help to confirm your answers?

**1791.** During Dr. Garner's days as a student last century, he often studied calculus in a dim unheated room with only one candle for light and heat. One particular day in mid-winter, after walking 10 miles (uphill both ways!) through knee-deep snow to attend class, he returned home too tired to study. After lighting the solitary candle on his desk, he walked directly away cursing his woeful situation. The temperature (in degrees Fahrenheit) and illumination (in percentage of candle-power) decreased as his distance (in feet) from his candle increased. In fact, he kept a record of this and in the table below is that information, just in case you may not believe the preceding sad tale!

<i>Distance</i> (feet)	<i>Temperature</i> (°F)	<i>Illumination</i> (% candle-power)
0	55.0	100
1	54.5	88
2	53.5	77
3	52.0	68
4	50.0	60
5	47.0	56
6	43.5	53

Assume that I get cold when the temperature is below 40°F and it is dark when the illumination is at most 50% of one candle-power.

- What is the average rate at which the temperature is changing when the illumination drops from 77% to 56%?
- I can still read my old unlit analog watch when the illumination is 64%. Can I still read my watch when I am 3.5 feet from the candle? Explain.
- Suppose that at 6 feet the instantaneous rate of change of the temperature is  $-4.5^\circ\text{F}$  per foot and the instantaneous rate of change of the illumination is  $-3\%$  candle-power per foot. Estimate the temperature and the illumination at 7 feet.
- Am I in the dark before I am cold or am I cold before I am in the dark? Explain.

## 9.5 Integrals... Again!

For use anytime after Section 4.9

EVALUATE.

$$1792. \int \frac{4y}{3y^2 + 2} dy$$

$$1793. \int \frac{3z - 4z^2 + 1}{\sqrt{z}} dz$$

$$1794. \int 3^{5y} dy$$

$$1795. \int \sin 7x dx$$

$$1796. \int x \sqrt[4]{3x^2 - 5} dx$$

$$1797. \int \sin^2 4x dx$$

$$1798. \int e^{\sin 2x} \cos 2x dx$$

$$1799. \int \sin 4x \cos^2 4x dx$$

$$1800. \int (\cos x - 2 \sin x) dx$$

$$1801. \int \frac{3x - x^3 + 1}{x^4} dx$$

$$1802. \int \frac{dx}{x}$$

$$1803. \int \frac{dx}{2x - 3}$$

$$1804. \int \cos(4x - 5) dx$$

$$1805. \int \frac{1}{3} x (3x^2 - 2)^4 dx$$

$$1806. \int 2^{3y^2} y dy$$

$$1807. \int \frac{\cos x}{\sin x - 3} dx$$

$$1808. \int \tan 2x dx$$

$$1809. \int \frac{e^{1/x}}{5x^2} dx$$

$$1810. \int \frac{e^{2x}}{e^{2x} - 7} dx$$

$$1811. \int \left( \frac{x^2}{3} - \frac{2x^4}{5} - \frac{3}{7} \right) dx$$

$$1812. \int (e^{2x} + 3)^5 e^{2x} dx$$

$$1813. \int \sqrt{x} (x^{1/3} - x^{2/5}) dx$$

$$1814. \int e^{\ln x^3} dx$$

$$1815. \int e^{5 \sin x} \cos x dx$$

$$1816. \int (3x - 2) \sqrt[5]{x} dx$$

$$1817. \int (x^2 + 3x - 2)^3 (2x + 3) dx$$

$$1818. \int \frac{x^2}{x^3 - 2} dx$$

$$1819. \int x^3 \sqrt{x^4 - 2} dx$$

$$1820. \int (x\sqrt{x} + 3\sqrt{x}) dx$$

$$1821. \int \left( \frac{\sqrt[3]{y}}{3} - \frac{3}{\sqrt[3]{y}} \right) dy$$

$$1822. \int (2x^{1/2} + 3x^{-1/2}) dx$$

$$1823. \int 5x \sqrt[3]{(x^2 + 1)^2} dx$$

$$1824. \int (5x - 4)^5 dx$$

## 9.6 Intégrale, Integrale, Integraal, Integral

For use anytime after Section 4.9

FIND ANTIDERIVATIVES OF THE FOLLOWING.

$$1825. \int \sec(2u) \tan(2u) \, du$$

$$1826. \int \cos^2(7u) \, du$$

$$1827. \int \cot(3x) \, dx$$

$$1828. \int e^{3x}(e^{3x} - 5)^5 \, dx$$

$$1829. \int x \sqrt[3]{3x^2 - 1} \, dx$$

$$1830. \int e \, dx$$

$$1831. \int (3u^2 - 2u^{-1/3}) \, du$$

$$1832. \int e^{\cos(3x)} \sin(3x) \, dx$$

$$1833. \int \frac{2}{x+3} \, dx$$

$$1834. \int 3^{2a} \, da$$

$$1835. \int 5 \cos(5x) \, dx$$

$$1836. \int \sin(4x) \, dx$$

$$1837. \int \frac{2x^2 + 3x - 2}{x} \, dx$$

$$1838. \int \frac{\sec^2 x}{5 + \tan x} \, dx$$

$$1839. \int 3x \sqrt[3]{3x^2 - 2} \, dx$$

$$1840. \int (8z + 16)^{11} \, dz$$

$$1841. \int \sqrt{x+2} \, dx$$

$$1842. \int \sin(6y) \, dy$$

$$1843. \int \cos(2x) \, dx$$

$$1844. \int \sec(2x) \, dx$$

$$1845. \int \cos(4x) \sin^5(4x) \, dx$$

$$1846. \int \cot a \, da$$

$$1847. \int 2 \cos(2x) \, dx$$

$$1848. \int 2x \sqrt{x-3} \, dx$$

**1849.** Using what you know about the derivatives of functions like  $e^x$ ,  $\ln x$ , and  $\sin x$ , find functions which satisfy the following equations. For parts (c) and (d), find two different functions.

a)  $y' - y = 0$

b)  $y' + y = 0$

c)  $y'' + y = 0$

d)  $y'' - y = 0$

---

Attaching significance to invariants is an effort to recognize what, because of its form or colour or meaning or otherwise, is important or significant and what is only trivial or ephemeral. A simple instance of failing in this is provided by the poll-man at Cambridge, who learned perfectly how to factorize  $a^2 - b^2$  but was floored because the examiner unkindly asked for the factors of  $p^2 - q^2$ . —H. W. Turnbull

## 9.7 Calculus Is an Integral Part of Your Life

For use anytime after Section 4.9

EVALUATE.

$$1850. \int \frac{4y}{(3y^2 + 2)^5} dy$$

$$1851. \int xe^{5x^2} dx$$

$$1852. \int \cos(5\theta) d\theta$$

$$1853. \int \cos^2(5\theta) d\theta$$

$$1854. \int \cos(5x) \sin^2(5x) dx$$

$$1855. \int \frac{2x}{5x^2 - 3} dx$$

$$1856. \int \frac{1}{3x} dx$$

$$1857. \int \sin(5\theta - 3\pi) d\theta$$

$$1858. \int 4^x \ln 5 dx$$

$$1859. \int x \cot(3x^2) dx$$

$$1860. \int \frac{e^{2x} - 7}{e^{2x}} dx$$

$$1861. \int 3^x \sin(3^x) dx$$

$$1862. \int (1 - \sin^4 \theta) \cos \theta d\theta$$

$$1863. \int \frac{1}{x-3} dx$$

$$1864. \int \frac{5}{2-3x} dx$$

$$1865. \int \frac{4y}{2-3y^2} dy$$

$$1866. \int \frac{5z^2}{1+2z^3} dz$$

$$1867. \int \frac{3z-2}{3z^2-4z} dx$$

$$1868. \int \frac{2 \sin(3\theta)}{1 + \cos(3\theta)} d\theta$$

$$1869. \int \frac{2 \cos(4\phi)}{3 - \sin(4\phi)} d\phi$$

$$1870. \int \frac{(\ln x)^4}{x} dx$$

$$1871. \int \frac{(\ln x)^3}{x} dx$$

$$1872. \int \sin^6 x \cos x dx$$

$$1873. \int \sin(3x) \cos^2(3x) dx$$

$$1874. \int \sin^2(4x) dx$$

$$1875. \int \sin^3(5x) dx$$

$$1876. \int \sin^3(2x) \cos^3(2x) dx$$

$$1877. \int \tan^2(2x) dx$$

$$1878. \int e^{2x} \tan^2(e^{2x}) dx$$

$$1879. \int \frac{1}{2x} dx$$

$$1880. \int_1^{10} \frac{10}{x^2} dx$$

$$1881. \int_1^4 2^x dx$$

$$1882. \int_2^3 \frac{(\sqrt{x} + 4)^{-1/2}}{2\sqrt{x}} dx$$

$$1883. \int_4^5 \frac{1}{(y+1)^5} dy$$

$$1884. \int_1^2 10e^x dx$$

$$1885. \int_{5/2}^{13/2} \frac{10}{(2x-4)^{3/2}} dx$$

$$1886. \int_0^1 2e^{5x} dx$$

$$1887. \int_0^5 e^{-0.25x} dx$$

$$1888. \int_1^2 14\sqrt{3x-1} dx$$

$$1889. \int_{-1}^0 2e^{-0.05x} dx$$

$$1890. \int_{-1}^1 \frac{5x}{(15+2x^2)^5} dx$$

## 9.8 Particles

*For use anytime after Section 4.14*

IN THE FOLLOWING PROBLEMS,  $s(t)$  IS POSITION,  $v(t)$  IS VELOCITY, AND  $a(t)$  IS ACCELERATION. FIND BOTH THE NET DISTANCE AND THE TOTAL DISTANCE TRAVELED BY A PARTICLE WITH THE GIVEN POSITION, VELOCITY, OR ACCELERATION FUNCTION.

**1891.**  $v(t) = e^{3t}$ , where  $0 \leq t \leq 2$

**1892.**  $s(t) = e^{-t+t^2}$ , where  $0 \leq t \leq 4$

**1893.**  $v(t) = \cos 2t$ , where  $0 \leq t \leq \pi$

**1894.**  $a(t) = 4t - 6$ , where  $0 \leq t \leq 2$  and  $v(0) = 4$

FIND THE AVERAGE VALUE OF EACH FUNCTION OVER THE GIVEN INTERVAL.

**1895.**  $H(x) = x^2 + x - 2$ ;  $[0, 4]$

**1896.**  $g(x) = 3e^{3x}$ ;  $[\ln 2, \ln 3]$

**1897.**  $R(x) = \sin x$ ;  $[0, 2\pi]$

**1898.**  $T(x) = \tan 2x$ ;  $[0, \frac{\pi}{8}]$

---

We speak of invention: it would be more correct to speak of discovery. The distinction between these two words is well known: discovery concerns a phenomenon, a law, a being which already existed, but had not been perceived. Columbus discovered America: it existed before him; on the contrary, Franklin invented the lightning rod: before him there had never been any lightning rod.

Such a distinction has proved less evident than it appears at first glance. Torricelli has observed that when one inverts a closed tube on a mercury trough, the mercury ascends to a certain determinate height: this is a discovery; but in doing this, he has invented the barometer; and there are plenty of examples of scientific results which are just as much discoveries as inventions. —*Jacques Hadamard*

## 9.9 Areas

*For use anytime after Section 4.16*

IN THE FOLLOWING EIGHT PROBLEMS, FIND THE AREA UNDER THE CURVE ON THE INTERVAL  $[a, b]$  BY USING

- A) A RIGHT-HAND RIEMANN SUM ON  $n$  EQUAL SUBINTERVALS;
- B) A LEFT-HAND RIEMANN SUM ON  $n$  EQUAL SUBINTERVALS;
- C) 2 TRAPEZOIDS ON EQUAL SUBINTERVALS;
- D) SIMPSON'S RULE WITH 2 PARABOLAS ON EQUAL SUBINTERVALS; AND
- E) A DEFINITE INTEGRAL.

**1899.**  $y = 8 - 3x$ ;  $[0, 2]$ ;  $n = 2$

**1900.**  $y = x^2$ ;  $[0, 2]$ ;  $n = 5$

**1901.**  $y = 3x^2 + 5$ ;  $[1, 4]$ ;  $n = 3$

**1902.**  $y = 7$ ;  $[-2, 6]$ ;  $n = 4$

FIND THE EXACT AREA OF THE REGION BOUNDED BY THE GIVEN CURVES.

**1903.**  $y = 25 - x^2$ ,  $y = 0$

**1904.**  $y = \sqrt{x - 2}$ ,  $y = 0$ ,  $x = 3$

**1905.**  $y = \frac{2}{x}$ ,  $y = 0$ ,  $x = 1$ ,  $x = e$

**1906.**  $y = \cos 2x$ ,  $y = 0$ ,  $x = 0$ ,  $x = \frac{\pi}{4}$

**1907.**  $y = (x - 1)(x - 2)(x - 3)$ ,  $y = 0$

**1908.**  $y = x^2$ ;  $y = x + 6$

**1909.**  $y = x^3 + 1$ ;  $y = 9$ ;  $x = 0$

**1910.**  $y = e^{3x}$ ;  $y = 8$ ;  $x = \ln 3$

**1911.**  $y = \tan x$ ;  $y = 1$ ;  $x = 0$

**1912.**  $y = 2x + 10$ ;  $y = x^2 + 2$ ;  $x = 0$ ;  $x = 4$

**1913.**  $y = x^3 - 3x - 3$ ;  $y = 5$ ;  $x = -2$ ;  $x = 2$

**1914.**  $y = (x + 2)^{3/2}$ ;  $y = -x$ ;  $x = 1$ ;  $x = 2$

**1915.**  $y = x$ ;  $y = x^2$

**1916.**  $y = x$ ;  $y = x^3$

**1917.**  $y = x^2$ ;  $y = x^3$

**1918.**  $y = \sin x$ ;  $y = \cos x$ ;  $y = 0$

**1919.**  $y = x^2 - 1$ ;  $y = 1 - x^2$

**1920.**  $y = x^2 - 2x - 3$ ;  $y = x - 4$

**1921.**  $y = x^2 - x - 15$ ;  $y = 10 - x$ ;  $y = 0$

**1922.**  $y = e^x \sqrt{e^x - 1}$ ;  $y = 0$ ;  $x = 0$ ;  $x = \ln 9$

**1923.**  $y = \csc x \cot x$ ;  $y = 0$ ;  $x = \frac{\pi}{6}$ ;  $x = \frac{\pi}{3}$



## 9.10 The Deadly Dozen

For use anytime after Section 5.2

**1924.** Find  $\frac{d}{dx} \int_{2x}^{-3} \sqrt{5t-3} dt$ .

**1925.** If  $F(x) = \int_7^x \frac{\ln(3\pi t)}{t} dt$ , then find  $F'(x)$ .

**1926.** Find the net distance and the total distance traveled by a particle from time  $t = 1$  to  $t = 3$  if the velocity is given by  $v(t) = -3t^2 + 7t - 2$ .

**1927.**  $\int \frac{1}{\sqrt{1-x^2}} dx = ?$

**1928.**  $\int \frac{1}{x \ln x} dx = ?$

**1929.** Let  $R$  be the region bounded by the  $x$ -axis, the  $y$ -axis, the line  $x = 4$  and the curve  $y = 2e^{3x}$ . Find the area of  $R$ .

**1930.** Find the area bounded by the lines  $y = x$ ,  $x = e$  and the hyperbola  $xy = 7$ .

**1931.** Find the area bounded by the  $x$ -axis and the curve  $y = 3x^3 - 6x^2 - 3x + 6$ .

**1932.** Find the average value of  $G(x) = \frac{\ln x}{x}$  over the interval  $[e, 2e]$ .

**1933.** Find the area of the region bounded by the curve  $y = x^2 - 3x^2 - 4x$  and the line  $y = 0$ .

**1934.** Find the average value of the function  $g(x) = 3\sqrt{x}$  over the interval  $[0, 2]$ .

**1935.** Let  $R$  be the region bounded by  $g(x) = 2/x$ ,  $x = 1$ ,  $x = 2$ , and  $y = 0$ .

- Approximate the area of  $R$  by using a left-hand Riemann sum with 2 subintervals.
- Use 2 trapezoids to approximate the area.
- Find the exact area of  $R$ .
- Find the volume of the solid generated by revolving  $R$  about the  $x$ -axis.
- Find the volume of the solid generated by revolving  $R$  about the  $y$ -axis.
- Set up an expression involving an integral that represents the perimeter of  $R$ .
- Use your calculator to evaluate the expression in part (f).

---

Mathematics is not only real, but it is the only reality. That is that entire universe is made of matter, obviously. And matter is made of particles. It's made of electrons and neutrons and protons. So the entire universe is made out of particles. Now what are the particles made out of? They're not made out of anything. The only thing you can say about the reality of an electron is to cite its mathematical properties. So there's a sense in which matter has completely dissolved and what is left is just a mathematical structure. —*Martin Gardner*

## 9.11 Two Volumes and Two Differential Equations

For use anytime after Section 5.6

- 1936.** Find the volume of the solid generated by revolving the region bounded by  $y = \arctan x$ ,  $y = \frac{\pi}{4}$ , and  $x = 0$  about the  $y$ -axis.
- 1937.** Let  $R$  be the region enclosed by the graph of  $y = 2e^{-x}$  and the line  $x = k$ .
- Find the area of  $R$  in terms of  $k$ .
  - Find the volume, in terms of  $k$ , of the solid generated when  $R$  is rotated about the  $x$ -axis.
  - What is the limit of the volume in part (b) as  $k \rightarrow \infty$ ?
- 1938.** Consider the differential equation  $y' = y - 2$  with initial condition  $y(0) = 1$ .
- Use Euler's method with 4 steps of size 0.2 to estimate  $y(0.8)$ .
  - Show that the exact solution of the differential equation is  $y = 2 - e^x$ .
  - Calculate the exact value of  $y$  when  $x = 0.8$ . How does this value compare with the approximation in part (a)?
- 1939.** The population of a variety of truffle increases at an annual rate equal to 3% of the current population  $P$ , measured in kilograms per acre; this takes into account the birth rate and death rate due to natural causes. Meanwhile humans harvest 0.5 kg/acre each year.
- Write a differential equation that describes the rate of change of the truffle population with respect to time, taking both the natural increase and the loss to human consumption into account.
  - If the current population is 10 kg/acre, use Euler's method with two steps to approximate the population four years from now.
  - Scarcity of the truffles leads to a call for halting the harvesting until the population reaches 12 kg/acre. What amount should then be harvested each year to exactly maintain this population?
  - Assume that the population is 10 kg/acre and there is no harvesting, so the truffles increase simply at a rate equal to 3% of the current population. Give a formula for  $P$  as a function of time. Then determine how long it will be until the population reaches 12 kg/acre.

---

In scientific thought we adopt the simplest theory which will explain all the facts under consideration and enable us to predict new facts of the same kind. The catch in this criterion lies in the word "simplest." It is really an aesthetic canon such as we find implicit in our criticisms of poetry or painting. The layman finds such a law as  $\frac{dx}{dt} = K \frac{d^2x}{dy^2}$  much less simple than "it oozes." The physicist reverses this judgment, and his statement is certainly the more fruitful of the two, so far as prediction is concerned. It is, however, a statement about something very unfamiliar to the plain man, namely the rate of change of a rate of change. —*John Burdon Sanderson Haldane*

## 9.12 Differential Equations, Part Four

For use anytime after Section 5.8

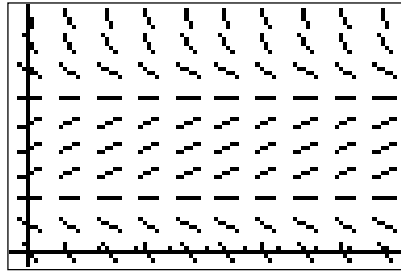
**1940.** A reasonable mathematical model for population of a species in a fixed region is that there are both a minimum and a maximum sustainable population. If the population is below the minimum, the species eventually dies out and becomes extinct. If the population is above the maximum, the population decreases, approaching the maximum as a limit.

- a) Assume the population of roadrunners in a region is given by

$$\frac{dP}{dt} = 0.05(P - 4)(12 - P),$$

where  $P$  is population in thousands and  $t$  is time in years. Show that the population is decreasing if  $P = 3$  and if  $P = 15$ , and increasing if  $P = 5$ .

- b) Below is the slope field for this differential equation. The  $x$ - and  $y$ -scale are both 2.



Explain why the slopes you calculated in part (a) are reasonable.

- c) Show that the population is stable if  $P = 4$  or if  $P = 12$ .
- d) On the slope field above, sketch the particular solution if  $P = 5$  when  $t = 0$ .
- e) Demonstrate your understanding of Euler's Method by using it starting at  $(0, 5)$  and calculating the next value of  $P$  with a step size  $\Delta t$  of 0.3.
- f) Use Euler's Method with  $\Delta t = 1$  year to calculate values of the particular solution in part (d). Record every even value of  $t$  with the corresponding value of  $P$  in a table.
- g) Plot the values from part (f) on the slope field above. How close did your solution in part (b) come to the numerical solution given by Euler's Method?
- h) Suppose that at time  $t = 8$ , the Game and Wildlife Commission brings 7000 more roadrunners into the region to help increase the population. Use the results of part (f) to get an initial condition, then sketch the graph of the population.
- i) Suppose the roadrunner population had been  $P = 3.99$  at time  $t = 0$ . Since this number is below the minimum sustainable population, the roadrunners are predicted to become extinct. Use Euler's Method with  $\Delta t = 1$  to estimate the year in which extinction occurs.
- j) Repeat part (i) using  $\Delta t = 0.1$ . Compare your answer to your answer in part (i). What conclusions can you draw from this?
- k) Find the particular solution  $P(t)$  assuming  $P(0) = 3.99$ . When does extinction occur?

### 9.13 More Integrals

For use anytime after Section 6.1

EVALUATE.

$$1941. \int \sqrt{x} \left( 3x - \frac{2}{x} + 5 \right) dx$$

$$1942. \int z dz$$

$$1943. \int 4x(5x^2 - 3)^{10} dx$$

$$1944. \int 5^{3x} \ln 7 dx$$

$$1945. \int 5te^{4t^2} dx$$

$$1946. \int x\sqrt{4x-1} dx$$

$$1947. \int \frac{3}{z} dz$$

$$1948. \int 3 \sin 2z dz$$

$$1949. \int 4 \cos 3z dz$$

$$1950. \int \frac{2w}{\sqrt{1-w^2}} dw$$

$$1951. \int x \left( 6x^2 + x + \frac{7}{x^4} \right) dx$$

$$1952. \int \frac{x}{x^2-1} dx$$

$$1953. \int e^{-x} dx$$

$$1954. \int \frac{e^{1/x}}{x^2} dx$$

$$1955. \int \frac{e^x}{e^x+1} dx$$

$$1956. \int \sin^2 x \cos x dx$$

$$1957. \int x \cot(x^2) dx$$

$$1958. \int \frac{x^2}{\sqrt[4]{x^3+2}} dx$$

$$1959. \int \frac{x+3}{\sqrt[3]{x^2+6x}} dx$$

$$1960. \int \frac{x^2+2x}{(x+1)^2} dx$$

$$1961. \int \frac{1-3y}{\sqrt{2y-3y^2}} dy$$

$$1962. \int \frac{2}{1+3u} du$$

$$1963. \int \frac{1}{1+4x^2} dx$$

$$1964. \int \frac{x}{(1+4x^2)^2} dx$$

$$1965. \int \frac{x^3-x-1}{(x+1)^2} dx$$

$$1966. \int \cos^2 x dx$$

$$1967. \int \frac{1}{\cos^2 3u} du$$

$$1968. \int \tan \theta d\theta$$

$$1969. \int \frac{\sin 2t}{1-\cos 2t} dt$$

$$1970. \int \frac{x-1}{x(x-2)} dx$$

$$1971. \int xe^{-x} dx$$

$$1972. \int \frac{\ln v}{v} dv$$

$$1973. \int e^{2 \ln u} du$$

## 9.14 Definite Integrals Requiring Definite Thought

*For use anytime after Section 6.1*

EVALUATE.

$$1974. \int_{-1}^1 (2x^2 - x^3) dx$$

$$1975. \int_{-3}^{-1} \left( \frac{1}{x^2} - \frac{1}{x^3} \right) dx$$

$$1976. \int_1^4 \frac{1}{\sqrt{y}} dy$$

$$1977. \int_{-2}^3 e^{-z/2} dz$$

$$1978. \int_{\pi/2}^{3\pi/2} \sin \phi d\phi$$

$$1979. \int_0^3 \frac{1}{\sqrt{1+w}} dw$$

$$1980. \int_4^8 \frac{q}{\sqrt{q^2-15}} dq$$

$$1981. \int_1^4 (1-x)\sqrt{x} dx$$

$$1982. \int_0^1 (x+1)e^{x^2+2x} dx$$

$$1983. \int_0^{\pi/4} \tan^2 x dx$$

$$1984. \int_0^{1/2} \frac{2x}{\sqrt{1-x^2}} dx$$

$$1985. \int_1^e \frac{1}{x} dx$$

$$1986. \int_1^2 \frac{1}{\sqrt[3]{t-1}} dt$$

$$1987. \int_0^{\pi/2} \frac{\sin \theta}{\sqrt{1-\cos \theta}} d\theta$$

$$1988. \int_{\pi/12}^{\pi/4} \frac{\cos(2\theta)}{\sin^2(2\theta)} d\theta$$

$$1989. \int_4^9 \frac{2+t}{\sqrt{t}} dt$$

$$1990. \int_0^1 \ln x dx$$

$$1991. \int_0^{\pi/6} \frac{\cos(2t)}{1+2\sin(2t)} dt$$

$$1992. \int_{\sqrt{2}}^2 \frac{u}{u^2-1} du$$

$$1993. \int_1^e x^2 \ln x dx$$

$$1994. \int_0^2 \ln(x^2+6) dx$$

$$1995. \int_1^2 (x^2 e^x - 2 \ln x) dx$$

$$1996. \int_0^{\ln 4} \frac{e^x}{\sqrt{e^{2x}+9}} dx$$

---

Numbers written on restaurant bills within the confines of restaurants do not follow the same mathematical laws as numbers written on any other pieces of paper in any other parts of the Universe. This single statement took the scientific world by storm. It completely revolutionized it. So many mathematical conferences got held in such good restaurants that many of the finest minds of a generation died of obesity and heart failure and the science of math was put back by years. —*Douglas Adams, Life, the Universe, and Everything*

## 9.15 Just When You Thought Your Problems Were Over...

For use anytime after Section 6.6

EVALUATE THE FOLLOWING INTEGRALS.

1997.  $\int_{-\infty}^0 10^x dx$

1999.  $\int_0^{\infty} ke^{-kx} dx$

1998.  $\int_0^{\infty} 10e^{-10x} dx$

2000.  $\int_1^{\infty} x^{-1/2} dx$

FIND THE AREA BETWEEN  $f$  AND THE  $x$ -AXIS ON THE INTERVAL GIVEN.

2001.  $f(x) = \frac{10}{x^2}; [1, \infty)$

2003.  $f(x) = 5e^{-5x}; [0, \infty)$

2002.  $f(x) = \frac{5}{x^{2/3}}; [1, \infty)$

2004.  $f(x) = k^{-kx}$ , for  $0 < k < 1$ ;  $[0, \infty)$

2005.  $f(x) = k^{-kx}$ , for  $k > 1$ ;  $[0, \infty)$

2006. The table below shows values of a continuous and differentiable function  $g(x)$ .

$x$	-4	-3	-2	-1	0	1	2	3	4
$g(x)$	0	0	12	27	36	30	0	-63	-168

Use the table to determine whether the integrals are positive, negative, or zero.

a)  $\int_{-3}^2 g(x) dx$

c)  $\int_2^2 g(x) dx$

e)  $\int_0^1 g(x+2) dx$

b)  $\int_{-4}^0 g(x) dx$

d)  $\int_2^{-1} g(x) dx$

f)  $\int_{-2}^1 g''(x) dx$

2007. Using the same table from problem 2006, compute the following integrals.

a)  $\int_{-1}^2 g'(x) dx$

c)  $\int_2^2 g'(x) dx$

b)  $\int_{-2}^0 g'(x) dx$

d)  $\int_3^{-1} 5g'(x) dx$

2008. Let  $H(x) = \int_{2x}^4 \frac{1}{1+t^2} dt$ . Find the value of  $c$  that satisfies the Mean Value Theorem for Derivatives for the function  $H(x)$  on the interval  $1 \leq x \leq 2$ .

**2009.** Let  $F(x)$  and  $F'(x)$  be defined by the following table.

$x$	-1	0	1	2	3	4	5
$F(x)$	18	0	-12	0	54	168	360
$F'(x)$	-15	-18	-3	30	81	150	237

- Find the equation of the tangent line to  $F(x)$  at  $x = -1$ .
- Find the equation of the normal line to  $F(x)$  at  $x = 2$ .
- Determine whether  $F(x)$  is increasing, decreasing, or neither at  $x = 0$ . Explain.
- Find the zeros of  $F(x)$  based on the information given.
- Find the  $y$ -intercept of  $F(x)$ .
- Find  $\int_2^3 F'(x) dx$ .
- Find  $\int_3^1 F''(x) dx$ .
- If  $H(x) = \int_{x^2}^2 F(t) dt$ , then find  $H'(1)$ .

**2010.** Find the equation of the normal line to the curve  $y = \int_{\pi/2}^x \sqrt[5]{\sin t} dt$  at the point on the curve where  $x = \pi/2$ .

**2011.** Find the average rate of change of  $g(x) = 3^{x^2}$  over the interval  $[0, 2]$ .

**2012.** Let  $R$  be the region bounded by  $f(x) = e^x$ ,  $g(x) = \ln x$ ,  $x = 1$ , and  $x = 2$ .

- Sketch the region  $R$ .
- Find the area of region  $R$ .
- Find the volume of the solid generated by revolving the region  $R$  around the  $x$ -axis.
- Find the volume of a solid whose base is  $R$  and the cross-sections of the solid perpendicular to the  $x$ -axis are squares.
- Find the volume of a solid whose base is  $R$  and the cross-sections of the solid perpendicular to the  $x$ -axis are semicircles.

**2013.** A particle moves along a straight line according to the position function  $x(t) = 1.5t^3 + 1.3t^2 - 2t + 3.4$ . Find the position of the particle when the particle is at rest.

## 9.16 Interesting Integral Problems

For use anytime after Section 6.6

**2014.** The following antiderivatives can be found by using clever variable substitutions or identities. Evaluate them.

$$\text{a) } \int \frac{1}{2 + \sin x} dx \quad \text{b) } \int_0^\pi (1 - \cos \theta)^5 \sin^3 \theta d\theta \quad \text{c) } \int \sec \theta \csc \theta d\theta$$

**2015.** Find the third-degree cubic polynomial function whose graph has an inflection point at  $(1, 3)$ , contains the point  $(2, 1)$ , and such that the area bounded by the graph between  $x = 1$  and  $x = 2$  above and the  $x$ -axis below is 3 square units.

**2016.** Find the area in the first quadrant bounded by the graph of  $y^2 = x^2 \left( \frac{a-x}{a+x} \right)$  for  $a > 0$ .

**2017.** The table at left shows four points on the function  $f$ .

$t$	$f(t)$
8	8.253
9	8.372
10	8.459
11	8.616

a) Estimate  $f'(10)$ .

b) Estimate  $f^{-1}(8.5)$

c) Estimate  $\int_8^{10} f'(t) dt$ .

**2018.** An experimental jet car runs along a track for 6 seconds before exploding into a giant fireball that could be seen for miles. From its starting time,  $t = 0$  seconds, its speed in feet per second is given by the formula  $V(t) = 0.08te^t$ . Answer the following questions, giving correct units.

a) How fast was it going at  $t = 6$  seconds?

b) What was its average acceleration over the first 6 seconds?

c) How far did it travel during the first 6 seconds?

**2019.** A particle moves along the  $s$ -axis so that its velocity at any time  $t \geq 0$  is given by  $v(t) = 3t^2 - 6t$ . The position at  $t = 2$  is  $s = 4$ .

a) Write an expression for the position  $s(t)$  of the particle at any time  $t \geq 0$ .

b) For what values of  $t$  for  $1 \leq t \leq 4$  is the particle's instantaneous velocity the same as its average velocity on the closed interval  $[1, 4]$ ?

c) Find the total distance traveled by the particle from time  $t = 1$  to  $t = 4$ .

---

Suppose we loosely define a religion as any discipline whose foundations rest on an element of faith, irrespective of any element of reason which may be present. Quantum mechanics for example would be a religion under this definition. But mathematics would hold the unique position of being the only branch of theology possessing a rigorous demonstration of the fact that it should be so classified. —*F. de Sua*



**2020.** Let the function  $f$  be defined on the interval  $[0, \pi]$  by  $f(x) = \int_0^x (1 + \cos t) dt$ .

- If they exist, determine the critical points of  $f$ .
- If possible, determine the minimum value of  $f$  on the interval  $[0, \pi]$ .

**2021.** A particle moves along the  $s$ -axis in such a way that at time  $t$ ,  $1 \leq t \leq 8$ , its position is given by  $s(t) = \int_1^t [1 - x \cos x - (\ln x)(\sin x)] dx$ .

- Write an expression for the velocity of the particle at time  $t$ .
- At what instant does the particle reach its maximum speed?
- When is the particle moving to the left?
- Find the total distance traveled by the particle from  $t = 1$  to  $t = 8$ .

**2022.** Oil has spilled into a straight river that is 100 meters wide. The density of the oil slick is  $d(x) = \frac{50x}{1+x^2}$  kilograms per square meter, where  $x \geq 0$  is the number of meters downstream from the spill. Assume the density of the oil slick does not vary from one shore to another and the approximate amount of oil in one strip of the river of width  $\Delta x$  is

$$(\text{density})(\text{area}) = d(x) \cdot 100\Delta x.$$

- Write a Riemann sum with 8 terms that approximates how much oil is within 80 meters of the source of the spill.
- Write a definite integral that gives the exact amount of oil that is within 80 meters of the source of the spill.
- Evaluate the integral in part (b).

**2023.** Let the function  $F$  be defined by  $F(x) = \int_0^x \frac{t^2 - 2t}{e^t} dt$ .

- Find  $F'(x)$  and  $F''(x)$ .
- If they exist, determine the critical points of  $F$ .
- Discuss the concavity of the graph of  $F$ .

---

I wanted certainty in the kind of way in which people want religious faith. I thought that certainty is more likely to be found in mathematics than elsewhere. But I discovered that many mathematical demonstrations, which my teachers expected me to accept, were full of fallacies, and that, if certainty were indeed discoverable in mathematics, it would be in a new field of mathematics, with more solid foundations than those that had hitherto been thought secure. But as the work proceeded, I was continually reminded of the fable about the elephant and the tortoise. Having constructed an elephant upon which the mathematical world could rest, I found the elephant tottering, and proceeded to construct a tortoise to keep the elephant from falling. But the tortoise was no more secure than the elephant, and after some twenty years of very arduous toil, I came to the conclusion that there was nothing more that I could do in the way of making mathematical knowledge indubitable. —*Bertrand Russell*

## 9.17 Infinitely Interesting Infinite Series

*For use anytime after Section 7.4*

DETERMINE IF EACH SERIES BELOW CONVERGES OR DIVERGES. IF IT IS GEOMETRIC AND CONVERGES, FIND ITS SUM.

2024.  $\sum_{n=1}^{\infty} \frac{3n-5}{5-2n}$

2030.  $\sum_{n=1}^{\infty} \left( e - \frac{1}{n^2} \right)$

2036.  $\sum_{n=1}^{\infty} \frac{2}{(3n+1)^{3/2}}$

2025.  $\sum_{n=1}^{\infty} \left( \frac{3}{2} \right)^{n-1}$

2031.  $\sum_{n=1}^{\infty} \left( 5 + \frac{3}{4n^2} \right)$

2037.  $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n}$

2026.  $\sum_{n=1}^{\infty} \ln \left( \frac{e^n}{n} \right)$

2032.  $\sum_{n=1}^{\infty} \frac{1}{1+2^n}$

2038.  $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^2}$

2027.  $\sum_{n=2}^{\infty} \left( \frac{2}{e} \right)^{n-1}$

2033.  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+2}}$

2039.  $\sum_{n=1}^{\infty} \frac{n}{e^n}$

2028.  $\sum_{n=1}^{\infty} \frac{2}{n+7}$

2034.  $\sum_{n=1}^{\infty} \frac{(n+1)!}{(n+3)!}$

2040.  $\sum_{n=1}^{\infty} 2^{-3n}$

2029.  $\sum_{n=1}^{\infty} \left( \frac{1}{4^n} - \frac{1}{5^n} \right)$

2035.  $\sum_{n=1}^{\infty} \frac{1}{n^2+4n+7}$

2041.  $\sum_{n=1}^{\infty} e^{-2n}$

2042. Respond to each of the following with “true” or “false.”

a)  $\sum_{n=1}^{\infty} \frac{1}{n^3}$  converges.

c)  $\sum_{n=1}^{\infty} \frac{1}{n \log n}$  converges.

b)  $\sum_{n=1}^{\infty} \frac{1}{n^{1+k}}$  converges, where  $k > 1$ .

d)  $\sum_{n=1}^{\infty} \frac{n+1}{(n+2)n!}$  converges.

2043. *Investigating the Logarithmic  $p$ -series*

a) Show that the integral below converges if and only if  $p > 1$ , where  $p$  is a positive constant.

$$\int_2^{\infty} \frac{1}{x(\ln x)^p} dx$$

b) How does the result in part (a) help in determining the convergence of the series below? Explain.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

## 9.18 Getting Serious About Series

*For use anytime after Section 7.5*

**2044.** Let  $a_n = \frac{1 + 2^n}{1 + 3(2^n)}$ .

a) Does  $\lim_{n \rightarrow \infty} a_n$  exist? If so, find it.

b) Does  $\sum_{n=0}^{\infty} a_n$  converge? Explain.

c) Does  $\sum_{n=0}^{\infty} (-1)^n a_n$  converge? Explain.

**2045.** Let  $a_n = \frac{2^n n + 1}{2^n n^2 + 1}$ .

a) Does  $\lim_{n \rightarrow \infty} a_n$  exist? If so, find it.

b) Does  $\sum_{n=0}^{\infty} a_n$  converge? Explain.

c) Does  $\sum_{n=0}^{\infty} (-1)^n a_n$  converge? Explain.

**2046.** If  $\sum_{n=1}^{\infty} 5 \left(\frac{3}{4}\right)^n$  converges, find its sum.

**2047.** Express the decimal  $0.516516516516\dots$  as a geometric series and find its sum.

**2048.** Find the interval of convergence of  $\sum_{n=0}^{\infty} \frac{(4x - 3)^{3n}}{8^n}$  and, within this interval, find the sum of the series as a function of  $x$ .

**2049.** Determine the values of  $x$  for which  $\sum_{n=1}^{\infty} \frac{2^n \sin^n x}{n^2}$  converges.

**2050.** Find the interval of convergence of  $\sum_{n=1}^{\infty} \frac{3^n (x - 2)^n}{2^n \sqrt{n + 2}}$ .

**2051.** A rubber ball rebounds to  $\frac{2}{3}$  of the height from which it falls. If it dropped from a height of 4 feet and is allowed to continue bouncing indefinitely, what is the total vertical distance it travels?

**2052.** Solve for  $x$ :  $\sum_{n=0}^{\infty} x^n = 20$ .

## 9.19 A Series of Series Problems

*For use anytime after Section 7.5*

**2053.** Determine the convergence or divergence of each series.

a)  $\sum_{n=2}^{\infty} \frac{(2n)!}{3^n(n-1)}$

b)  $\sum_{n=1}^{\infty} \frac{n^2 + 3n - 4}{n!}$

c)  $\sum_{n=1}^{\infty} \left(1 + \frac{1}{2n}\right)^{3n}$

**2054.** Find the radius of convergence of each series.

a)  $\sum_{n=0}^{\infty} \frac{(5x)^n}{3^n}$

b)  $\sum_{n=1}^{\infty} \frac{n^2(2x-3)^n}{6^n}$

c)  $\sum_{n=0}^{\infty} \frac{\ln(n+2)x^{n+2}}{(n+2)!}$

**2055.** Which of the following describes the interval of convergence for  $\sum_{n=0}^{\infty} \frac{(x^3 - 2)^{2n}}{4^n}$ ?

A)  $-\sqrt[3]{4} < x < \sqrt[3]{4}$

D)  $0 \leq x < \sqrt[3]{4}$

B)  $-\sqrt[3]{2} < x < \sqrt[3]{6}$

E)  $0 < x \leq \sqrt[3]{4}$

C)  $0 < x < \sqrt[3]{4}$

**2056.** The series  $\sum_{n=0}^{\infty} \frac{(-1)^n(2n-1)!}{n^5}$

A) converges absolutely

B) converges conditionally

C) diverges

**2057.** The series  $\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$

A) converges absolutely

B) converges conditionally

C) diverges

**2058.** The radius of convergence of  $\sum_{n=0}^{\infty} \frac{(2x-5)^n}{n!}$  is

A) 0

B)  $\frac{1}{2}$

C) 1

D) 2

E)  $\infty$

## CHAPTER 10

---

# GROUP INVESTIGATIONS

---

## A Note About the Group Investigations

### *Some Suggestions*

The problems in this chapter are to be completed in groups consisting of between two and four persons. The final report and class presentation that presents your solution is the ultimate result of collaboration, knowledge, cleverness, and diligence. The report must be an organized and neatly-produced document. Complete sentences, good grammar, proper spelling, and correct mathematics are expected.

To help you take full advantage of this Investigation, the following suggestions are offered.

- *Get started immediately!* You will not be able to complete the investigation on a last minute basis. Portions of the project will move slowly and working in a group requires more time due to scheduling difficulties.
- Read over the entire investigation carefully before you begin discussing or completing any portion of it.
- Initially, you may not know how to begin. Don't panic! A discussion with other group members usually generates some ideas.
- The procedure for a complete investigation is not as clear as it is for solving standard homework problems. You will possibly need to make assumptions in order to simplify the problem. Justify these assumptions and comment on how they may or may not affect the final result.
- If any questions persist or there is a lack of clarity on some point, be certain to discuss them with Dr. Garner *before* writing the final report.

## 10.1 Finding the Most Economical Speed for Trucks

A trucking company would like to determine the highway speed that they should require of their drivers. The decision is to be made purely on economical grounds, and the two primary factors to be considered are driver wages and fuel consumption. Wage information is easily obtained: drivers earn from \$11.00 to \$15.00 an hour, depending on experience. Incorporating the fuel consumption question is much more difficult and the company has hired you as consultants in order to solve the problem for them. Correctly assuming that fuel consumption is closely related to fuel economy at various highway speeds, they have provided you with the following statistics taken from a U. S. Department of Transportation study:

*Miles per gallon at selected speeds*

Vehicle	50 mph	55 mph	60 mph	65 mph
Truck #1	5.12	5.06	4.71	*
Truck #2	5.41	5.02	4.59	4.08
Truck #3	5.45	4.97	4.52	*
Truck #4	5.21	4.90	4.88	4.47
Truck #5	4.49	4.40	4.14	3.72
Truck #6	4.97	4.51	4.42	*

\* Due to laws controlling fuel injection, this vehicle could not be operated at 65 mph.

The company expects a written report with your recommendation. The report should include justification for your conclusion.

*Note:* Although the report should include the mathematical detail of your work, the overall presentation should not assume the reader has had experience with the techniques you employ.

## 10.2 Minimizing the Area Between a Graph and Its Tangent

Given a function  $f$  defined on  $[0, 1]$ , for which of its non-vertical tangent lines  $T$  is the area between the graph of  $f$  and  $T$  minimal? Develop an answer for three different nonlinear functions of your own choosing. Choose no more than one function from a particular class of functions (i.e., polynomial, radical, rational, trigonometric, exponential, logarithmic). Carefully explain the reasoning leading to your conclusions. Looking back at your results, try to formulate and then verify any conjectures or generalizations they suggest. (*Hint:* Stick to functions whose concavity doesn't change on  $[0, 1]$ .)

## 10.3 The Ice Cream Cone Problem

You are to place a sphere of ice cream into a cone of height 1.

- (1) What radius of the sphere will give the most volume of ice cream inside the cone (as opposed to above the cone) for a cone with a base angle of  $30^\circ$ ?
- (2) What percent of this sphere of most volume lies inside the cone?

You will need to be thoughtful in choosing the variable which will best help you answer this question. When you have determined the optimal radius, be sure to make an accurate drawing of your sphere in the cone to insure the reasonableness of your result.

## 10.4 Designer Polynomials

For this project, imagine that you are a calculus instructor needing some “nice” examples of function for your class. Your purpose is to help the students understand the relationships between polynomial functions and their graphs. In particular, you are interested in quadratic functions,  $x^2 + cx + d$ , cubic functions,  $x^3 + bx^2 + cx + d$ , and quartic functions,  $x^4 + ax^3 + bx^2 + cx + d$ , where  $a$ ,  $b$ ,  $c$ , and  $d$  have integer values. Thoroughly answer the following questions, giving complete justification for each result.

- A. Quadratic functions of the form  $f(x) = x^2 + cx + d$ .
- (1) How many critical points are there for each choice of  $c$  and  $d$ ?
  - (2) Are the critical points maxima, minima, or neither?
  - (3) How should you choose  $c$  and  $d$  to insure that all critical points occur at integer values of  $x$ ?
  - (4) What must the shape of the graph of  $f$  be?
- B. Cubic functions of the form  $g(x) = x^3 + bx^2 + cx + d$ .
- (5) How many inflection points are there?
  - (6) Produce examples where the inflection points and critical points all occur at integer values of  $x$  where  $g(x)$ : (a) 0 critical points, (b) 1 critical point, (c) 2 critical points.
  - (7) Write general rules for choosing  $b$ ,  $c$ , and  $d$  to produce families of examples in question 6.
- C. Quartic functions of the form  $h(x) = x^4 + ax^3 + bx^2 + cx + d$ .
- (8) Choose  $a$ ,  $b$ ,  $c$ , and  $d$  so that  $h(x)$  has 3 critical points at integer values of  $x$ . Produce some general techniques to generate a family of examples with 3 critical points, all at integer values of  $x$ . (*Hint*: Rather than solving cubic equations to determine if there are 3 integer-valued roots, start with the roots and produce the cubic equation. For example, given the roots 2, 3, and  $-1$ , the equation is  $(x - 2)(x - 3)(x + 1) = 0$  or  $x^3 - 4x^2 + x + 6 = 0$ .)

## 10.5 Inventory Management

A computer services firm regularly uses many cartons of computer paper. They purchase the cartons in quantity from a discount supplier in another city at a cost of \$22.46 per carton, store them in a rented warehouse near company grounds and use the paper gradually as needed. There is some confusion among company managers as to how often and in what quantity paper should be ordered. On one hand, since the supplier is providing out-of-town delivery by truck, there is a basic \$360 charge for every order regardless of the number of cartons purchased, assuming the order is for no more than 3000 cartons (the truck’s capacity). The cost has been used by some managers as an argument for placing large orders as infrequently as possible. On the other hand, as other managers have argued, large orders lead to large warehouse inventories and associated costs of at least two kinds that should be considered. First, they claim, whatever money is used to pay for paper that will only sit in the warehouse for a long time could instead, for a while at least, be allocated to some profit producing activity. At the very least such



money could be accumulating interest in a bank account. This loss of investment opportunity and associated earnings is referred to as the “opportunity cost” resulting from the investment in paper inventory. Secondly, the company has to pay rent for the warehouse. While other company property is stored there as well, the managers agree that a fraction of the rent equal to the fraction of the warehouse space occupied by paper should be viewed as part of the cost of storing paper. These latter two costs, collectively referred to as the inventory “holding cost” and estimated to be 18 cents per carton per week, have been used to justify claims by some that paper orders should be smaller and placed more frequently. In hopes of resolving the confusion, the managers have hired you as a consultant.

After talking more with various company personnel, asking many questions and inspecting company records, you have found the following summary notes. Use them along with additional modeling and analysis as the basis for a report to the managers recommending in what quantity and how often paper should be ordered.

*Note 1* Company data on the number of cartons used per week is shown below. No one thinks the usage rate will change in any significant way.

Week	Cartons	Week	Cartons	Week	Cartons	Week	Cartons	Week	Cartons	Week	Cartons
1	150	10	152	19	149	28	150	37	151	46	152
2	149	11	150	20	150	29	147	38	148	47	151
3	150	12	149	21	150	30	152	39	150	48	148
4	151	13	149	22	150	31	150	40	150	49	150
5	153	14	150	23	150	32	151	41	149	50	150
6	150	15	150	24	152	33	148	42	152	51	151
7	148	16	150	25	150	34	151	43	150	52	147
8	150	17	151	26	148	35	150	44	149		
9	150	18	150	27	151	36	150	45	147		

*Note 2* We should not let the paper supply run out. Managers agree that a work stoppage would be disastrous for customer relations, so paper would be purchased from a local source rather than allowing a stoppage to occur. The best local price is \$46.90 per carton compared to \$22.46 from the usual discount supplier.

*Note 3* The discount supplier is *very* reliable about providing quick delivery. When an order is placed in the morning, she has never failed to deliver before 5 PM the same day. It seems safe to count on this. So in modeling, for simplicity, we can assume that the new order arrives just as the last stored carton is used.

*Note 4* Managers seem to agree that the goal in deciding how much and how often to order should be to minimize average weekly cost associated with the purchase and storage of paper. Average weekly cost has three constituents: purchase cost ( $\$22.46 \times$  number of cartons ordered per week), delivery cost ( $\$360 \times$  number of orders per week), and holding cost. When pushed to be more precise about holding cost, manager consensus was that average holding cost per week should be measured as 18 cents per carton per week times the average inventory between orders (i.e., the average number of cartons stored in the warehouse from the time one order arrives to the time the next order arrives).

*Note 5* It will simplify modeling and analysis to assume that the inventory level (the number of boxes stored) and time are continuous variables.

## 10.6 Optimal Design of a Steel Drum

A 55-gallon tight head steel drum is constructed by attaching 18 gage (0.0428 inches thick) steel disks to the top and bottom of a cylinder created by rolling up a 20 gage (0.0324 inches thick) steel sheet. (See Figure 1.)

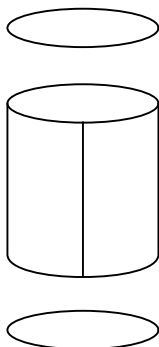


Figure 1

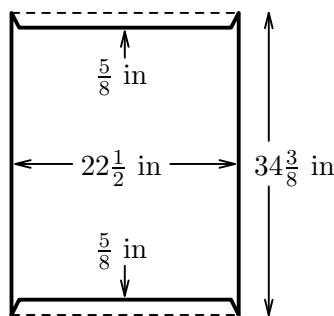


Figure 2

The vertical seam on the cylinder is welded together and the top and bottom are attached by a pressing-sealing machine. The pressing-sealing process requires approximately  $\frac{13}{16}$  inches from the cylinder and  $\frac{3}{4}$  inches from the disk to be curled together and hence these inches are lost in the final dimensions. Additionally, the top and bottom are set down  $\frac{5}{8}$  inches into the cylinder. For clarification, refer to Figure 2, a copy of the American National Standard (ANSI) specification diagram.

Steel can be purchased in coils (rolls) of any specified width. Construction costs can be summarized as follows.

18 gage steel is 45 cents per square foot  
 20 gage steel is 34 cents per square foot  
 welding and pressing-sealing cost 10 cents per foot  
 cutting steel costs 2 cents per foot

Is the ANSI-specified drum the most efficient use of material in order to obtain the required 57.20 gallon minimum volume capacity of a 55 gallon drum? Fully justify your answer.

---

1) Write an exam for this course. 2) Take it. —*Dr. David Mumford's final exam for his mathematics courses at Harvard*

# CHAPTER 11

---

## **CALCULUS LABS**

---

## A Note About the Labs

### *The Lab Session*

For you to have a successful lab, the following suggestions and guidelines are offered.

- Read the lab *carefully* and *intelligently*. Every sentence carries directions, information, or both.
- If work is requested “Before the Lab,” do it with understanding before you and your partner work together.
- While “In the Lab,” think as you go through the steps of the lab. Always ask yourself and your partner if the results make sense.
- Discriminate between incidental and relevant results. Record important results and calculations for inclusion in your lab report.
- The work requested “After the Lab” is an extension of the ideas developed during the lab. This can be completed by yourself or with your partner.

### *The Lab Report*

The lab report should be thoughtful, well-written, and a neatly organized document that summarizes your work “Before the Lab,” your experiences “In the Lab,” your explorations “After the Lab,” and what you learned as a result of the experience.

A rubric for grading your lab is on the next page. This rubric gives specific details for the types of things you should include in your report.

Of all the parts of the lab report listed on the rubric, the one that is graded the most harshly is the *conclusion*. The conclusion is your opportunity to show that you understood the purpose of the lab, saw patterns in the data, and gained significant insights. Be as general in your conclusions as you dare, but back them up by specific references to your data and calculations.

As for the rest of the lab report, thoroughly answering the questions asked in the lab should be sufficient to receive full credit for the other items. Remember to sketch the graphs as needed, show the work that goes into equations that you solve, and answer all questions completely and accurately.

There is no need for you and your partner to submit separate lab reports; one from the pair of you is all that is required.

There is no need to type your lab report. However, if you have the Equation Editor plug-in for MS-Word, or if you know  $\text{\LaTeX}$ , you may submit a typed lab report. The only advantage to typing a lab report is that you may submit it by e-mail by midnight on the due date, rather than turning it in during class. (Warning: scanned, hand-written lab reports submitted by e-mail are not acceptable.)

## Calculus Lab Rubric

**Heading:** names, lab number, lab title

- 1: complete
- 0: incomplete

**Answers:** completeness of responses to all questions; severity of flaws in reasoning

- 3: complete; no reasoning flaws
- 2: complete; minor reasoning flaws
- 1: mostly complete; multiple reasoning flaws
- 0: incomplete; serious reasoning flaws

**Ideas Communicated:** demonstrates understanding of the problems, ideas, and mathematics

- 3: complete understanding
- 2: substantial understanding
- 1: some understanding
- 0: no understanding

**Mathematics:** computations, formulas, accuracy of results

- 3: complete; no errors
- 2: complete; few minor errors
- 1: mostly complete; multiple minor errors
- 0: incomplete; major errors

**Graphs, Tables, Charts:** organization, appropriate usage, correct labels and details

- 3: clear, relevant; labeled, detailed
- 2: clear, relevant; little or no labels or details
- 1: unclear or inappropriate; no labels, no details
- 0: no graph, table, or chart present

**Conclusion:** clear and concise, addresses the purpose of the lab and states the knowledge gained

- 6: complete response
- 3: unclear or partial grasp of knowledge gained
- 0: no evidence of purpose or knowledge

**Mechanics:** grammar, spelling, punctuation, neatness

- 2: neat; few minor errors
- 1: sloppy; few errors
- 0: sloppy; multiple errors

Your grade is the total points earned multiplied by 2, for a maximum of 42 points possible.

## Lab 1: The Intermediate Value Theorem

### Goals

- To discover and acquire a feel for one of the major theorems in calculus
- To apply the theorem in both practical and theoretical ways
- To understand why continuity is required for the theorem

### In the Lab

This lab will motivate you to discover an important general theorem of calculus. The theorem, called the Intermediate Value Theorem (IVThm for short), has a clear geometric interpretation but requires a careful mathematical formulation. This lab will develop your intuition for both aspects of the theorem.

1. Consider the function  $f$  defined on the closed interval  $[1, 2]$  by  $f(x) = x^5 - 4x^2 + 1$ .
  - (a) Compute the numerical values of  $f(1)$  and  $f(2)$ . Without graphing, explain why you think the graph of  $f$  must cross the  $x$ -axis somewhere between  $x = 1$  and  $x = 2$ . That is, why must there be a number  $c$  between 1 and 2 such that  $f(c) = 0$ ?
  - (b) Plot the graph of the function  $f$  on  $[1, 2]$  to support what you said in part (a) above. Also, estimate any zeros of  $f$  (points at which  $f(c) = 0$ ) on  $[1, 2]$ .
  
2. Consider the function  $g(x) = \begin{cases} x^2 + 1 & -3 \leq x \leq 0 \\ 1 - x^2 - x^4 & 0 < x \leq 2 \end{cases}$  defined on the closed interval  $[-3, 2]$ .
  - (a) Compute the numerical values of  $g(-3)$  and  $g(2)$ . Explain why you think the graph of  $g$  must cross the  $x$ -axis somewhere between  $x = -3$  and  $x = 2$ .
  - (b) Plot the graph of  $g$  on  $[-3, 2]$  and estimate any zeros of  $g$ .
  - (c) Now replace  $g$  by the slightly altered function  $h(x) = \begin{cases} x^2 + 1 & -3 \leq x \leq 0 \\ -1 - x^2 - x^4 & 0 < x \leq 2 \end{cases}$  defined on  $[-3, 2]$ .  
Once again evaluate the function at its endpoints and think about whether it will take on the value 0 somewhere in the interval. Also plot the graph, and try to explain in general terms why you came to your conclusion.
  
3. We are now ready to formulate a statement of the Intermediate Value Theorem. Based upon the observations above, fill in the blanks to complete the following.

Given a \_\_\_\_\_ function  $f$  defined on the closed interval  $[a, b]$  for which 0 is between \_\_\_\_\_ and \_\_\_\_\_, there exists a point  $c$  between \_\_\_\_\_ and \_\_\_\_\_ such that \_\_\_\_\_.

Note: The above statement of the IVThm is an example of what is called an “existence theorem.” It says that a certain point exists, but does not give a rule or algorithm for how to find it. In the applications of the theorem, the key fact is that such a point exists, not its specific value.

4. Use the IVThm to prove that the function defined by  $f(x) = \sin(x^2) - \cos x$  has a zero on the interval  $[0, 1]$ , where  $x$  is in radians. Use graphing and/or root finding methods to estimate a zero on the interval.
5. If your oven is at  $250^\circ$  and you turn it off, is there ever an instant when the oven temperature is  $170^\circ$ ? Explain your answer and its relation to the IVThm.
6. If you remove marbles from a bag one at a time, must there always come a time when the bag contains half the number of marbles with which it began? Again, explain your answer and its relation to the IVThm.
7. In your notebook, sketch the graph of a continuous function  $g$  over the interval  $[0, 1]$ . Now draw the graph of another continuous function  $h$  over the same interval with the property that  $h(0) < g(0)$  and  $h(1) > g(1)$ .
  - (a) Must these two graphs cross? Express this behavior in terms of a condition involving the functions  $g$  and  $h$  and a point  $c$  in the interval  $(0, 1)$ .
  - (b) Give a proof that what you observed in part (a) must always be true for any two continuous functions  $g$  and  $h$  on  $[0, 1]$  with the property that  $h(0) < g(0)$  and  $h(1) > g(1)$ .  
(*Hint*: Consider the new function  $f$  defined by  $f(x) = h(x) - g(x)$ .)
  - (c) One plate has been in the freezer for a while, the other is in a warm oven. The locations of the two plates are then switched. Will there be a moment when the plates are at the same temperature? How does your answer relate to the ideas developed in parts (a) and (b) above?
8. A fixed point of a function  $f$  is a point  $c$  in the domain of  $f$  for which  $f(c) = c$ .
  - (a) In your notebook, sketch the graph of a continuous function  $f$  defined on an interval  $[a, b]$  and whose values also lie in the interval  $[a, b]$ . You may choose the endpoints  $a$  and  $b$  of the interval randomly, but be sure that the range of the function is contained within its domain. Locate a point  $c$  on your graph that is a fixed point of  $f$ .
  - (b) Try to draw the graph of a continuous function  $f$  defined of an interval  $[a, b]$  with values in the interval  $[a, b]$  that has no fixed points. What is getting in your way? Prove that any continuous function  $f$  defined on  $[a, b]$  with values in  $[a, b]$  must have a fixed point  $c$  in the interval  $[a, b]$ .  
*Hint*: Use your pictures and draw in the graph of the line  $y = x$ .

## Lab 2: Local Linearity

### Goals

- To define the slope of a function at a point by zooming in on that point
- To see examples where the slope, and therefore the derivative, is not defined

### In the Lab

You learned in Algebra I that the slope of a line is  $\frac{y_2 - y_1}{x_2 - x_1}$ , or  $\frac{\Delta y}{\Delta x}$ , where  $(x_1, y_1)$  and  $(x_2, y_2)$  are any two points on the line. Most functions we see in calculus have the property that if we pick a point on the graph of the function and zoom in, we will see a straight line.

1. Graph the function  $f(x) = x^4 - 10x^2 + 3x$  for  $-4 \leq x \leq 4$ . Zoom in on the point  $(3, 0)$  until the graph looks like a straight line. Pick a point on the curve other than the point  $(3, 0)$  and estimate the coordinates of this point. Calculate the slope of the line through these two points.

The number computed above is an *approximation to the slope of the function*  $f(x) = x^4 - 10x^2 + 3x$  at the point  $(3, 0)$ . This slope is also called the *derivative* of  $f$  at  $x = 3$ , and is denoted  $f'(3)$ . (The mark  $'$  is called a *prime* mark, and “ $f'(3)$ ” is read “ $f$  prime of 3.”)

2. Use zooming to estimate the slope of the following functions at the specified points.
  - (a)  $f(x) = x^4 - 6x^2$  at  $(1, -5)$
  - (b)  $f(x) = \cos x$  at  $(0, 1)$
  - (c)  $f(x) = \cos x$  at  $(\frac{\pi}{2}, 0)$
  - (d)  $f(x) = (x - 1)^{1/3}$  at  $(2, 1)$

So far in this lab you have used the graph of a function to estimate the value of its derivative at a specified point. Sometimes, however, a function does not have a slope at a point and therefore has no derivative there.

3. Graph  $f(x) = (x - 1)^{1/3}$  again. This time, zoom in on  $(1, 0)$ . Describe what you see. By examining your graphs, explain why the slope is undefined at  $x = 1$ . For this function, conclude that  $f'(1)$  does not exist.
4. Find the zeros of the function  $f(x) = x^4 - 6x^2$  and make a note of these points. Then graph the function  $f(x) = |x^4 - 6x^2|$ . By looking at the graph and zooming in on its zeros and other points you select, decide at which points the function  $f$  has a derivative and at which points it does not. Support your answers with appropriate sketches.



**After the Lab**

5. The derivative of  $f$  at a point  $a$  is defined verbally as “the limit of the slope of  $f$  at points  $(a, f(a))$  and  $(a + h, f(a + h))$  as  $a + h \rightarrow a$ .” It is defined analytically by the formula

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}.$$

Why is “ $a + h \rightarrow a$ ” the same as “ $h \rightarrow 0$ ”? Explain in your own words how calculating the slope of a function at the point  $(a, f(a))$  by repeated zooming is related to the computation of the derivative  $f'(a)$  by this analytic definition.

6. Use zooming to investigate  $f'(0)$ , the slope of the curve  $y = f(x)$  at  $x = 0$ , for the following two functions.

$$(a) f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$(b) f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

## Lab 3: Exponentials

### Goals

- To investigate the derivative of exponential functions
- To define an exponential function as a limit and as a series

### In the Lab

1. Exponential functions are functions of the form  $f(x) = a^x$ . They play an important role in the application of mathematics. In a calculus course, we naturally wonder how to find  $f'(x)$ . Experiments with something as simple looking as  $f(x) = 2^x$  quickly show that first guesses are generally wrong.

- (a) Demonstrate the absurdity of thinking that  $g(x) = x2^{x-1}$  might be the derivative of  $f(x) = 2^x$ . For example, have your calculator plot the graphs of these two functions. Think about what the graph of  $2^x$  would look like if its derivative were negative. Also look at  $x = 0$  where  $x2^{x-1}$  changes sign.
- (b) We need to go back to the definition of the derivative to derive a formula for  $f'(x)$  where  $f(x) = 2^x$ . Supply reasons for each of the following three steps:

$$\frac{d}{dx}(2^x) = \lim_{h \rightarrow 0} \frac{2^{x+h} - 2^x}{h} = \lim_{h \rightarrow 0} \frac{2^x(2^h - 1)}{h} = 2^x \lim_{h \rightarrow 0} \frac{2^h - 1}{h}.$$

Use your calculator to gather evidence that  $\lim_{h \rightarrow 0} \frac{2^h - 1}{h}$  really exists; in other words, find the approximate value of this limit. You might evaluate the quotient for small values of  $h$  or zoom in on the graph of the quotient near the  $y$ -axis. Notice that the derivative of  $f$  is simply this constant times the function itself.

- (c) Modify the process in part (b) to find a formula for the derivative of  $f(x) = 3^x$ . Notice again that the derivative is a multiple of  $f$ , but with a different constant factor.
- (d) Show in general with any positive number  $a$  as the base of an exponential function  $f(x) = a^x$ , that

$$f'(x) = \frac{d}{dx}(a^x) = a^x \lim_{h \rightarrow 0} \frac{a^h - 1}{h}.$$

- (e) We are interested in locating a value of  $a$  for which the constant  $\lim_{h \rightarrow 0} \frac{a^h - 1}{h}$  is equal to 1. Such a base will give an exponential function whose derivative is exactly itself. Check back to parts (b) and (c) to see that the base  $a = 2$  gives a constant less than 1, while  $a = 3$  gives a constant greater than 1. Try to narrow in on a value of  $a$  between 2 and 3 that gives an exponential function whose derivative is itself.

The number you have discovered in problem 1(e) is commonly designated by  $e$ . With this notation, your discovery in problem 1(e) is that if  $f(x) = e^x$ , then  $f'(x) = e^x$ .

2. (a) If you deposit money in a savings account paying interest at an annual rate of  $r$ , your deposit will grow by a factor of  $1 + r$  after one year. (If the interest rate is 5%, then we use  $r = 0.05$ .) If, however, the bank compounds the interest in the middle of the year, your deposit will grow by a factor of  $1 + \frac{r}{2}$  after the first six months and another factor of  $1 + \frac{r}{2}$  after the second six months. Thus the total growth is  $(1 + \frac{r}{2})^2$ . Verify that  $(1 + \frac{r}{2})^2$  is slightly more than  $1 + r$  for all nonzero values of  $r$ .
- (b) By what factor will your deposit grow in one year if the bank compounds quarterly? If it compounds monthly? Daily? Hourly? The limiting value of the yearly growth factor as the number of compounding periods increases to infinity is  $\lim_{n \rightarrow \infty} (1 + \frac{r}{n})^n$ . This is the growth factor used if the bank compounds continuously.
- (c) Substituting  $r = 1$  into the limit in part (b) gives  $\lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n$ . Use your calculator to approximate the value of this limit. Where have you seen this number before?
- (d) In part (b) we introduced the limit  $\lim_{n \rightarrow \infty} (1 + \frac{r}{n})^n$ , which is a function of  $r$ . You evaluated this function for  $r = 1$  in part (c). If  $r = 2$ , we have  $\lim_{n \rightarrow \infty} (1 + \frac{2}{n})^n$ . Use your calculator to evaluate this limit. Compare your answer with the number  $e^2$ . Repeat for  $r = 5$ , comparing your answer with the number  $e^5$ . In fact the function  $e^r$  can be defined by this limit for all values of  $r$ , namely,  $e^r = \lim_{n \rightarrow \infty} (1 + \frac{r}{n})^n$ .
3. Consider the infinite series  $1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$ . The exclamation mark in the denominators indicates the factorial of a positive integer. For example,  $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ .
- (a) Substitute  $x = 1$  into this series and sum the first 8 terms. What number do you obtain? Now substitute  $x = 2$  into the series. Where have you seen this number before?
- (b) Compute the derivative of the series (term by term). How does this result relate to the result of problem 1(e)? What function of  $x$  do you think this series represents? Give as much evidence as you can for your answer.

### After the Lab

4. We now know the derivative of  $e^x$  is  $e^x$ —but what about functions like  $e^{5x}$  or  $e^{x^3}$ ? Use the chain rule to determine the derivatives of the following.
- (a)  $e^{3x}$
- (b)  $e^{x^2}$
- (c)  $e^{3x^2-1}$
- (d)  $e^{\sin x}$

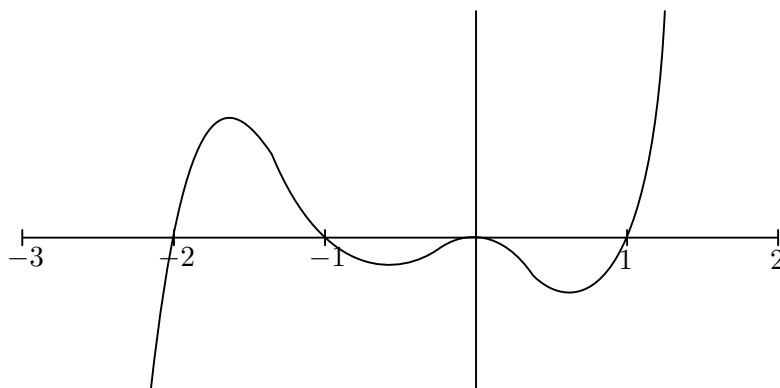
## Lab 4: A Function and Its Derivative

### Goals

- Given the graph of the function, to be able to visualize the graph of its derivative

### Before the Lab

In this lab, you will be asked to compare the graph of a function like the one in the figure below to that of its derivative. This exercise will develop your understanding of the geometric information that  $f'$  carries.



You will need to bring an example of such a function into the lab with you, one whose graph meets the  $x$ -axis at four or five places over the interval  $[-3, 2]$ . During the lab, your partner will be asked to look at the graph of your function and describe the shape of its derivative (and you will be asked to do the same for your partner's function). One way to make such a function is to write a polynomial in its factored form. For example,  $f(x) = x^2(x - 1)(x + 1)(x + 2)$  is the factored form of the function in the figure above. Its zeros are at 0, 1,  $-1$ , and  $-2$ .

- Give another example of a polynomial  $g$  of degree at least 5 with four or five real zeros between  $-3$  and  $2$ . You will use this polynomial in problem 4.
  - Your polynomial:  $g(x) =$
  - Its zeros:
  - Its derivative:  $g'(x) =$

### In the Lab

- Let  $f(x) = x^2(x - 1)(x + 2) = x^5 + 2x^4 - x^3 - 2x^2$ .
  - Find the derivative of  $f$ . Plot the graphs of both  $f$  and  $f'$  in the same window on your calculator over the interval  $-2.5 \leq x \leq 1.5$ .

Answer the following questions by inspection of this graph:

- (b) Over what intervals does the graph of  $f$  appear to be rising as you move from left to right?
  - (c) Over what intervals does the graph of  $f$  appear to be above the  $x$ -axis?
  - (d) Over what intervals does the graph of  $f$  appear to be falling as you move from left to right?
  - (e) Over what intervals does the graph of  $f$  appear to be below the  $x$ -axis?
  - (f) What are the  $x$ -coordinates of all the maximums and minimums of the graph of  $f$ ?
  - (g) For what values of  $x$  does the graph of  $f'$  appear to intersect the  $x$ -axis?
3. Let  $f(x) = \frac{x}{1+x^2}$ .
- (a) Find the derivative of  $f$ . Plot the graphs of both  $f$  and  $f'$  in the same window on your calculator over the interval  $-3 \leq x \leq 3$ .
  - (b) Answer the same set of questions as in parts (b)–(g) above.
4. On the basis of your experience so far, write a statement that relates where a function is rising, falling, and has a high point or low point to properties you have observed about the graph of its derivative.
5. Now let  $g$  be the function that your lab partner brought into the lab. (If you have no partner, just use your own function). In this problem you will use your statement from problem 4 to predict the shape of the graph of  $g'$ , given only the shape of the graph of  $g$ .
- (a) Have your lab partner produce a plot of the graph of  $g$  over the interval  $-3 \leq x \leq 2$ . Your partner may need to adjust the height of the window to capture maxima and minima. On the basis of this plot, use your conjecture to imagine the shape of the graph of  $g'$ . In particular, find where  $g'$  is above, where  $g'$  is below, and where  $g'$  meets the  $x$ -axis. Carefully sketch a graph of both  $g$  and your version of  $g'$  on your paper, labeling each graph.
  - (b) Now have your lab partner plot the graph of  $g'$  on a calculator. Add a sketch of the actual graph of  $g'$  onto your drawing. Compare your graph with the calculator drawn graph. How did you do?
  - (c) Using the function you wrote in problem 1, do parts (a) and (b) again.
6. Consider the function  $f(x) = |x^2 - 4|$ . A graph of the function will help you answer these questions.
- (a) There are two values of  $x$  for which the derivative does not exist. What are these values, and why does the derivative not exist there?
  - (b) Find the derivative of  $f$  at those values of  $x$  where it exists. To do this recall that  $f$  can be defined by

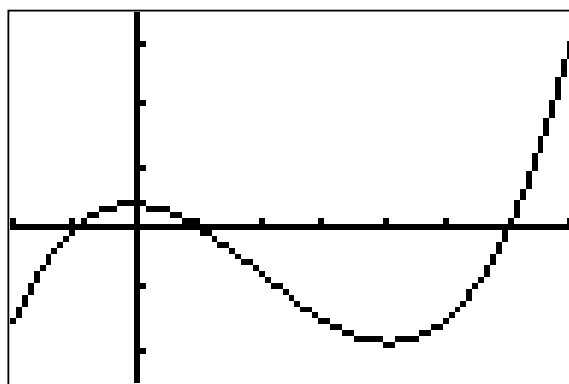
$$f(x) = \begin{cases} x^2 - 4 & -2 \geq x, x \geq 2 \\ 4 - x^2 & -2 \leq x \leq 2 \end{cases}$$

You can compute the derivative for each part of the definition separately.

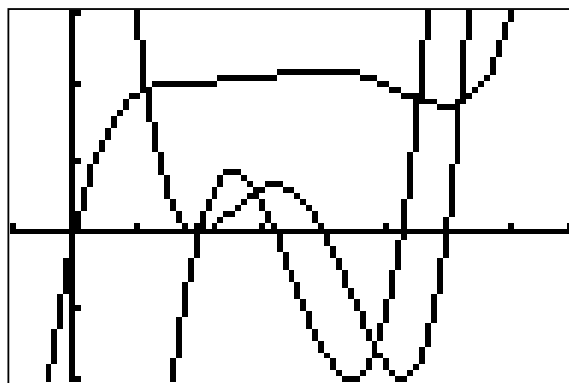
- (c) Give a careful sketch of  $f$  and  $f'$  (disregarding the places where  $f$  is not defined) over the interval  $-4 \leq x \leq 4$ . Does your conjecture from problem 4 still hold? Do you need to make any modifications?

### After the Lab

7. Consider the function  $f(x) = 2^x$ . Some people think that  $f'(x) = x2^{x-1}$ . On the basis of your conjecture, explain why this cannot be true.
8. This lab has given you experience in using what you know about the shape of the graph of a function  $f$  to visualize the shape of its derivative function  $f'$ . What about going backwards? Suppose that your partner had given you the graph of  $f'$ . Would you be able to reconstruct the shape of the graph of  $f$ ? If  $f'$  is positive, for example, does your conjecture enable you to rule out certain possibilities for the shape of  $f$ ? The graph in the figure below is that of the derivative of  $f$ . Use your conjecture to construct a possible graph for the function of  $f$  itself. The important part of this problem is neither the actual shape that you come up with, nor its position in the  $xy$ -plane, but your reasons for choosing it. Why isn't there a unique function that has  $f'$  for its derivative?



9. The figure below shows the graphs of three functions. One is the position of a car at time  $t$  minutes, one is the velocity of that car, and one is its acceleration. Identify which graph represents which function and explain your reasoning.



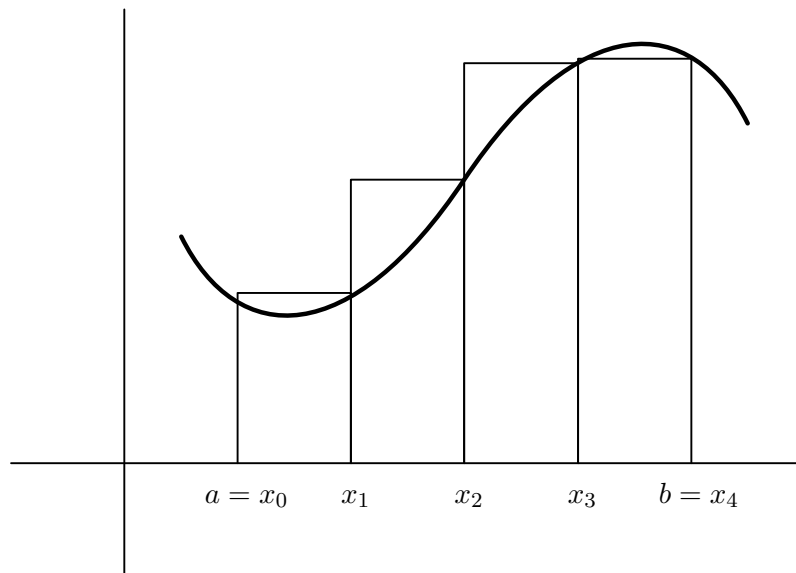
## Lab 5: Riemann Sums and Integrals

### Goals

- To approximate the area under a curve by summing the areas of coordinate rectangles
- To develop this idea of Riemann sums into a definition of the definite integral
- To understand the relationship between the area under a curve and the definite integral

### Before the Lab

1. (a) The figure below shows the graph of a function  $f$  on the interval  $[a, b]$ . We want to write an expression for the sum of the areas of the four rectangles that will depend only upon the function  $f$  and the interval endpoints  $a$  and  $b$ . The four subintervals that form the bases of the rectangles along the  $x$ -axis all have the same length; express it in terms of  $a$  and  $b$ . How many subinterval lengths is  $x_2$  away from  $a = x_0$ ? Write expressions for  $x_1$ ,  $x_2$ ,  $x_3$  and  $x_4$  in terms of  $a$  and  $b$ . What are the heights of the four rectangles? Multiply the heights by the lengths, add the four terms, and call the sum  $R(4)$ .



- (b) Generalize your work in part (a) to obtain an expression for  $R(n)$ , the sum of rectangular areas when the interval  $[a, b]$  is partitioned into  $n$  subintervals of equal length and the right-hand endpoint of each subinterval is used to determine the height of the rectangle above it. Write your expression for  $R(n)$  using summation notation. In order to do this, first figure out a formula for  $x_k$ , the right-hand endpoint of the  $k$ th subinterval. Then check that your formula for  $x_k$  yields the value  $b$  when  $k$  takes on the value  $n$ .
- (c) Let  $L(n)$  denote the sum of rectangular areas when left-hand endpoints rather than right-hand endpoints are used to determine the heights of the rectangles. Add some details to the figure above to illustrate the areas being summed for  $L(4)$ . What

modification in the expression for  $R(n)$  do you need to make in order to get a formula for  $L(n)$ ?

- (d) Check your formulas with Dr. Garner!

### In the Lab

2. Consider the function  $f(x) = 3x$  on the interval  $[1, 5]$ .
- (a) Apply your formulas for  $R(4)$  and  $L(4)$  to this function. In your notebook, sketch a picture to illustrate this situation. Use geometry to determine the exact area  $A$  between the graph of  $f$  and the  $x$ -axis from  $x_1$  to  $x_5$ .
  - (b) It is possible to make the computations easier by using a calculator program. To learn how to use this program, complete TI-Lab #5 on page 286. You are allowed to use the program to complete the remainder of this lab.
  - (c) Now use your calculator program to get values for  $R(n)$  and  $L(n)$  for  $n = 40$ . How are the sizes of  $R(40)$  and  $L(40)$  related to the area  $A$ ? Explain this relationship and make a conjecture about what should happen for larger and larger values of  $n$ . Test your conjecture with a few more values of  $n$  of your own choosing and record your results in a table.
3. Consider the function  $f(x) = \frac{1}{\sqrt{x}}$  on the interval  $[0.1, 10]$ . Plot the graph of  $f$  and use the ideas developed above to approximate the area  $A$  under the graph of  $f$  and above the  $x$ -axis over the given interval. What is the relation among  $R(n)$ ,  $L(n)$ , and the area  $A$  now? Explain any difference that you see between the situation here and the situation in problem 2.
4. Now let  $f(x) = 4 - x^2$  on the interval  $[-2, 2]$ . Again, plot the graph and estimate the area under it on this interval. How are the values of  $R(n)$  and  $L(n)$  related this time? What has changed and why? Does this prevent you from getting a good estimate of the area? Explain!
5. So far we have worked with functions that give nonnegative  $y$ -values on the interval. There is, however, nothing in the formulas you have developed that depends on nonnegative  $y$ -values. Here we consider what happens geometrically when the function takes on *negative* values.
- (a) Consider the function  $f(x) = 3x$  on the interval  $[-4, 2]$ . In your notebook, sketch the graph of  $f$  on this interval and include appropriate rectangles for computing  $R(3)$  and  $L(3)$ . On subintervals where the function is negative, how are the areas of the rectangles combined in obtaining the overall values for  $R(3)$  and  $L(3)$ ? What value do  $R(n)$  and  $L(n)$  seem to approach as  $n$  increases? How can you compute this value geometrically from your sketch?
  - (b) Now consider the function  $f(x) = 3x^2 - 2x - 14$  on the interval  $[-2, 3]$ . Use your calculator to plot it. Determine the value that  $R(n)$  and  $L(n)$  seem to approach. Explain with the help of your graph why you think this is happening.

Here is a summary of the two important points so far:



- Approximation by rectangles gives a way to find the area under the graph of a function when that function is nonnegative over the given interval.
- When the function takes on negative values, what is approximated is not an actual area under a curve, though it can be interpreted as sums and differences of such areas.

In either case, the quantity that is approximated is of major importance in calculus and in mathematics. We call it the *definite integral* of  $f$  over the interval  $[a, b]$ . We denote it by

$$\int_a^b f(x) \, dx.$$

For well-behaved functions it turns out that we can use left-hand endpoints, right-hand endpoints, or any other points in the subinterval to get the heights of the approximating rectangles. Any such sum of areas of approximating rectangles (over any partition of  $[a, b]$  into subintervals, equal in length or not) is called a Riemann sum. We obtain the definite integral as a limit of the Riemann sums as the maximum subinterval length shrinks to 0. In particular, for sums based on right-hand endpoints and equal length subintervals, we have

$$\lim_{n \rightarrow \infty} R(n) = \int_a^b f(x) \, dx.$$

Similarly,

$$\lim_{n \rightarrow \infty} L(n) = \int_a^b f(x) \, dx.$$

- 6.** For each of the definite integrals below, use either left or right-hand Riemann sums to approximate its value. Make your own decision about what values of  $n$  to use. Then use the definite integral function on your calculator to obtain another approximation. Comment on the degree to which the values agree.

(a)  $\int_0^1 e^x \, dx$

(b)  $\int_1^3 (x^3 - 3x^2 - 2x + 3) \, dx$

(c)  $\int_{-1}^3 \sin(x^2) \, dx$

## Lab 6: Numerical Integration

### Goals

- To understand the geometry behind two methods of numerical integration: the Trapezoid Rule and Simpson's Rule
- To gain a feel for the relative speeds of convergence of Riemann sums, Trapezoid Rule and Simpson's Rule

### Before the Lab

Suppose we want to calculate the definite integral  $\int_a^b f(x) dx$ . The Fundamental Theorem of Calculus states that

$$\int_a^b f(x) dx = F(b) - F(a),$$

where  $F$  is an antiderivative of  $f$ . Some integrals are easily evaluated this way; for example,

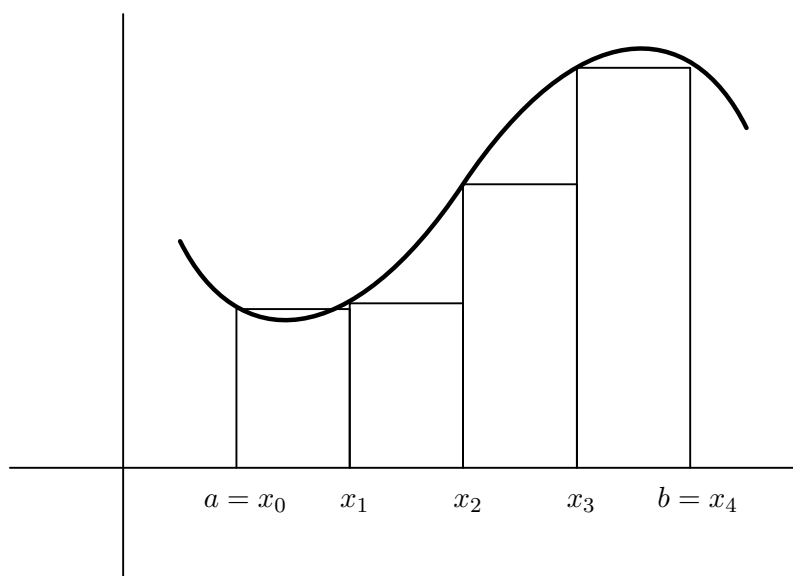
$$\int_0^{\pi/2} \cos x dx = \sin \frac{\pi}{2} - \sin 0 = 1.$$

However, integrals such as

$$\int_0^{\pi/2} \cos \sqrt{x} dx \quad \text{and} \quad \int_0^1 \sqrt{1+9x^4} dx$$

still give us problems since we cannot find usable expressions for the antiderivatives of the integrands. In this lab we will explore several methods for computing numerical approximations to such integrals.

**Riemann Sums.** One approach that you have seen in the definition of an integral is to form a Riemann sum. In this method, we replace the area under the curve  $y = f(x)$ ,  $a \leq x \leq b$ , by the area of some rectangles. In the figure below, we have a picture of a Riemann sum using four subintervals of equal length, with the height of each rectangle being the value of the function at the left-hand endpoint of that subinterval.



1. In this lab, we will be comparing several numerical answers for the value of

$$\int_0^1 (5x^4 - 3x^2 + 1) dx$$

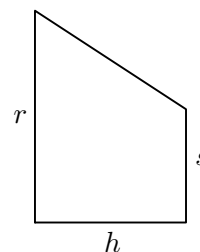
with the exact answer obtained by direct integration. What is the exact answer to this definite integral?

2. Make a table to hold your answers for problems 2a, 2b, 2c, 3d, and 5a. Your table should include the method used, the number of subintervals, the approximation to the integral, the error in the approximation, and the width of a subinterval.
- Use a Riemann sum with  $n = 4$  subintervals (show all work—do not use the calculator program) to approximate  $\int_0^1 (5x^4 - 3x^2 + 1) dx$ . Be sure to specify whether you used a left-hand or right-hand sum.
  - Use a the Riemann sum program on your calculator to approximate the same integral with  $n = 16$  subintervals.
  - Using calculator experimentation, find a value for  $n$  so that the Riemann sum gives an answer that is accurate to 0.001.

### In the Lab

**Trapezoid Rule.** In Riemann sums, we replace the area under the curve by the area of rectangles. However, the corners of the rectangles tend to stick out. Another method is to form trapezoids instead of rectangles. We will now develop the formula for the sum of the area of these trapezoids. This formula is known as the Trapezoid Rule.

3. (a) The formula for the area of a trapezoid is easy to derive. Divide the trapezoid in the figure at right into a rectangle and a right triangle. The area of the rectangle is \_\_\_\_\_. The area of the triangle is \_\_\_\_\_. Show the algebra necessary to get the total area to be  $\frac{h}{2}(r + s)$ .

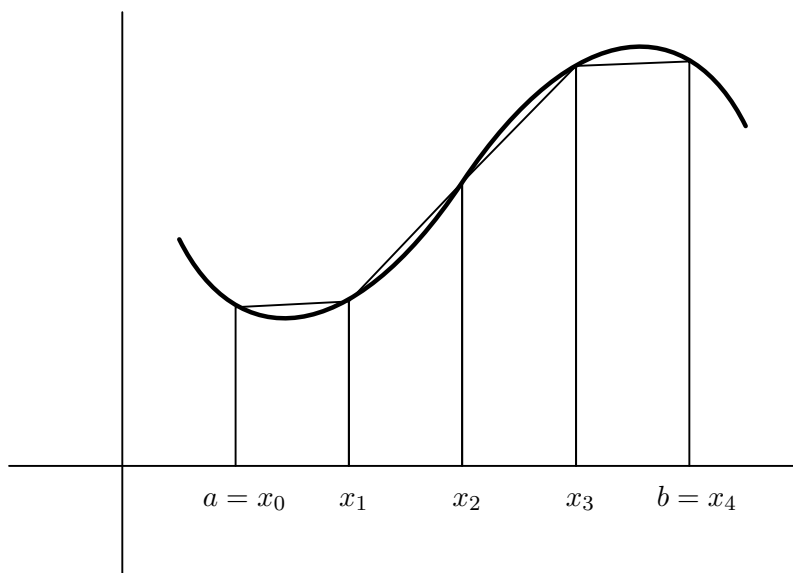


- (b) Apply this formula four times to the four trapezoids in the figure on the next page. Let  $T_4$  denote the sum of the areas of the four trapezoids. Show the algebra necessary to get that

$$T_4 = \frac{\Delta x}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)],$$

$$\text{where } \Delta x = \frac{b - a}{4} = \frac{x_4 - x_0}{4}.$$

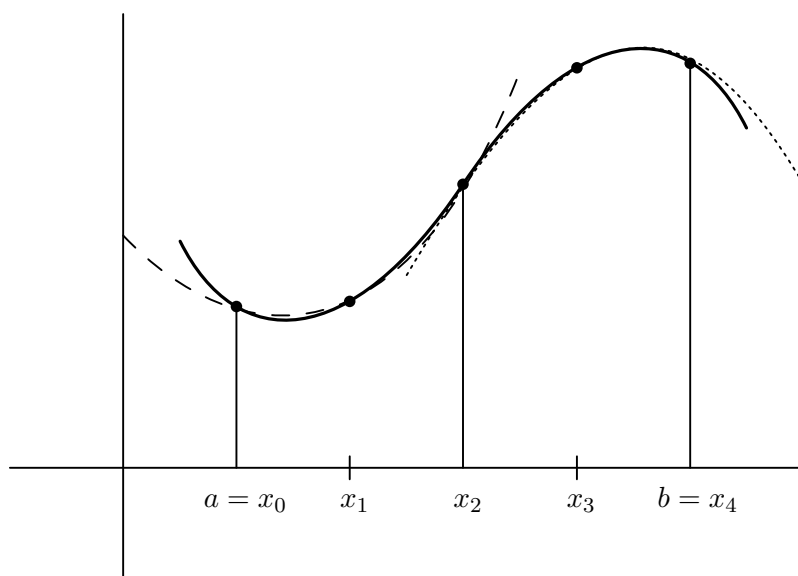
- (c) If we use  $n$  equally spaced subintervals instead of 4, then we have that  $T_n$  is the sum of the areas of the  $n$  trapezoids. Derive a formula for  $T_n$ .
- (d) Repeat problem 2 using the Trapezoid Rule, putting your data in the table. *NOTE:* Now it would be a good idea to read and complete *TI Lab 6: Approximating Integrals with Sums II* on page 287. You are free to use the calculator programs described in that TI Lab to complete this Lab.



**Simpson's Rule.** In the Trapezoid Rule, we replaced pieces of the curve by straight lines. In Simpson's Rule, we replace pieces of the curve by parabolas. To approximate  $\int_a^b f(x) dx$ , we divide  $[a, b]$  into  $n$  equally spaced subintervals, where  $n$  is even. Simpson's Rule relies on the fact that there is a unique parabola through any three points on a curve. A picture of Simpson's Rule where  $n = 4$  is given below. The dashed line is the parabola through the three points  $(x_0, f(x_0))$ ,  $(x_1, f(x_1))$  and  $(x_2, f(x_2))$ , while the dotted line is the parabola through the three points  $(x_2, f(x_2))$ ,  $(x_3, f(x_3))$  and  $(x_4, f(x_4))$ . The details are messy (so take my word for it!), but the area under the first parabola through  $(x_0, f(x_0))$ ,  $(x_1, f(x_1))$  and  $(x_2, f(x_2))$  is given by

$$\frac{\Delta x}{3} [f(x_0) + 4f(x_1) + f(x_2)],$$

where  $\Delta x = \frac{b-a}{4} = \frac{x_4-x_0}{4}$ .



4. (a) Let  $S_4$  be the sum of the areas under the  $\frac{4}{2} = 2$  parabolas in the figure above. Show the algebra necessary to get the formula for  $S_4$ :

$$S_4 = \frac{\Delta x}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)].$$

- (b) Let  $S_n$  be the sum of the areas under  $\frac{n}{2}$  parabolas. Write a formula for  $S_n$ .
5. (a) Repeat problem 2 using Simpson's Rule, again using the table to organize your data.  
 (b) What value of  $n$  did you need for each method to get the answer to the desired accuracy? Which method needed the smallest value of  $n$ ? (We call this the fastest method.) Which needed the largest? (This is the slowest method.)
6. The previous problems have been artificial, since we are easily able to compute the integral exactly. As stated in the beginning, we often use numerical integration when we cannot find an antiderivative. Let us now investigate

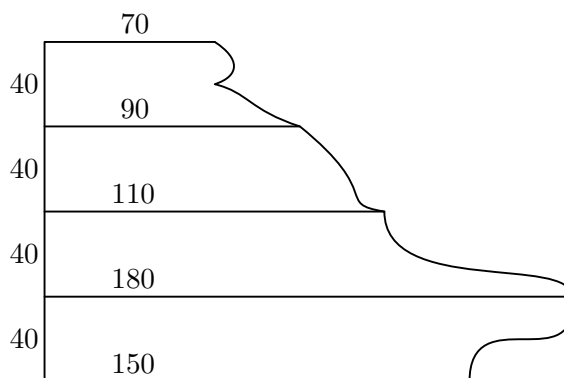
$$\int_0^1 \sqrt{1 + 9x^4} dx,$$

an integral that we cannot find exactly. This integral arises in calculation of the length of the curve  $y = x^3$ , where  $0 \leq x \leq 1$ .

- (a) Approximate the integral above using both the Trapezoid Rule and Simpson's Rule. Experiment with different values of  $n$  until you are convinced that your answers are accurate to 0.001.
- (b) How did you decide to stop? How do you get a feel for the accuracy of your answer if you do not have the exact answer to compare it to? (Do *not* compare to the value given by the calculator's `fnInt` command.)
- (c) Which method seems the fastest?

**Functions Given by Data.** Problem 6 illustrates the use of numerical integration to approximate  $\int_a^b f(x) dx$  when it is difficult or impossible to find an antiderivative for  $f$  in terms of elementary functions. In applications it is often the case that functions are given by tables or by graphs, without any formulas attached. For these functions, we only know the function value at specified points. Numerical integration is ideally suited for integrating this type of function. Notice that in this situation we cannot possibly find an antiderivative.

7. A map of an ocean front property is drawn below, with measurements in meters. What is its area?



8. The data given below describe the velocity  $v(t)$  of the \$249,000 1992 Lamborghini Diablo at time  $t$  seconds. Let  $x(t)$  denote the distance the car travels at time  $t$ ,  $0 \leq t \leq 10$ . Find  $x(10)$ . Discuss what method you used and how good you think your answer is.

$t$	$v(t)$
0	0 mph
1	14 mph
2	27 mph
3	40 mph
4	53 mph
5	64 mph
6	70 mph
7	77 mph
8	84 mph
9	90 mph
10	96 mph

### After the Lab

9. If  $f''(x) > 0$  for all  $x$  in  $[a, b]$ , would the value given by the Trapezoid rule be larger or smaller than the actual value of  $\int_a^b f(x) dx$ ? Why?
10. *Extra Credit* If  $h(x)$  is a linear function, it is easy to see that the Trapezoid rule is exactly equal to the area under  $h(x)$ ; that is,

$$T_1 = \int_a^b h(x) dx.$$

Similarly, if  $g(x)$  is quadratic, Simpson's rule gives the exact area under  $g(x)$ ; that is,

$$S_2 = \int_a^b g(x) dx.$$

It is surprising, however, that Simpson's rule also gives the exact area under a *cubic* function even though the approximating function is not an exact fit!\* Show that the following are exactly equal:

$$S_2 = \int_0^1 x^3 dx.$$

---

\* Simpson's rule is still of interest to mathematicians. In the February 2006 issue of *The American Mathematical Monthly*, L. A. Talman proved that Simpson's rule gives the exact area under fifth-degree polynomials.

## Lab 7: Indeterminate Limits and l'Hôpital's Rule

### Goals

- To recognize limits of quotients that are indeterminate
- To understand l'Hôpital's Rule and its applications
- To appreciate why l'Hôpital's Rule works

### Before the Lab

In this lab, we are interested in finding limits of quotients in cases which are referred to as indeterminate. This occurs, for example, when both the numerator and the denominator have a limit of 0 at the point in question. We refer to this kind of indeterminacy as the  $\frac{0}{0}$  case.

One of the standard limit theorems allows us to compute, under favorable conditions, the limit of a quotient of functions as the quotient of the limits of functions. More formally,

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$$

provided limits for  $f$  and  $g$  exist at  $a$  and  $\lim_{x \rightarrow a} g(x) \neq 0$ .

1. For four of the limits below, use the result stated above to evaluate the limits without the help of a calculator. For each of the other two, explain why the limit theorem does not apply, say what you can about the limit of the quotient, and indicate which are indeterminate in the  $\frac{0}{0}$  sense described above.

(a)  $\lim_{x \rightarrow 2} \frac{x^2 + 1}{x - 4}$

(b)  $\lim_{x \rightarrow 0} \frac{e^x - 1}{\sin(x) + 3}$

(c)  $\lim_{x \rightarrow 2} \frac{\sqrt{x^2 + 5}}{\sin x}$

(d)  $\lim_{x \rightarrow 2} \frac{1 - \frac{2}{x}}{\sin(\pi x)}$

(e)  $\lim_{x \rightarrow 3} \frac{x^2 - 9}{\sqrt{1 + x}}$

(f)  $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$  (Assume  $f$  is differentiable at  $a$ .)

### In the Lab

2. (a) Plot the function  $y = \frac{\ln x}{x^2 - 1}$  and, from the graph, determine or estimate the value of  $\lim_{x \rightarrow 1} \frac{\ln x}{x^2 - 1}$ .

- (b) Repeat the procedure suggested in part (a) to obtain the limit for problem 1(d).

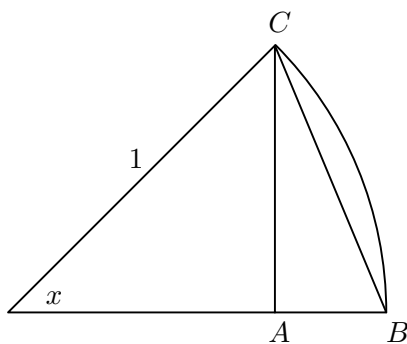
If indeterminate quotients were always such a specific nature and we were able and willing to use graphical or numerical estimates, there would be little need to pursue these matters further. Often, however, indeterminates occur in more abstract and general situations. Thus we seek a correspondingly general approach that will apply to a wide variety of indeterminate situations.

3. The result we will be exploring is known as l'Hôpital's Rule. One version of it says that if  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$  is indeterminate, but  $f$  and  $g$  both have derivatives at  $a$  with  $g'(a) \neq 0$ , then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

- (a) By taking the appropriate derivatives, apply the above result to the quotient  $\frac{\ln x}{x^2 - 1}$  at the point  $a = 1$ . Also check the result graphically by plotting, on the same axes, the quotient of the functions  $f$  and  $g$  defined by  $f(x) = \ln x$  and  $g(x) = x^2 - 1$  along with the quotient of their derivatives (not the derivative of the quotient!) as called for by l'Hôpital's Rule. Does  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$ ?
- (b) Do the same as above for the quotient of functions given in problem 1(d).
4. (a) Use l'Hôpital's Rule to compute  $\lim_{x \rightarrow 0} \frac{1 - e^{3x}}{\sin(x) + x}$ .
- (b) Let  $f$  be a function that is differentiable at  $a$ . Perhaps the most famous indeterminate of all is  $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$ . What does l'Hôpital's Rule say in this situation? Are you surprised?
- (c) Make a conjecture about continuing the procedure called for in l'Hôpital's Rule in situations where both  $f'(a)$  and  $g'(a)$  are also 0. Apply your conjecture to compute  $\lim_{x \rightarrow 0} \frac{x - \sin x}{1 - \cos x}$ .

5. Consider the sector of a unit circle with angle  $x$  (in radians) as pictured in the figure below. Let  $f(x)$  be the area of the triangle  $ABC$ , while  $g(x)$  is the area of the curved shape  $ABC$  (the segment of the circle plus the triangle).



- (a) By thinking geometrically, try to make a guess about the limit of  $f(x)/g(x)$  as the angle  $x$  approaches 0.



- (b) Show that  $f(x) = \frac{1}{2}(\sin x - \sin x \cos x)$  and  $g(x) = \frac{1}{2}(x - \sin x \cos x)$ .
- (c) Using the result stated in part (b), compute the actual limit you guessed at in part (a).

**After the Lab**

6. The  $\frac{0}{0}$  case is not the only indeterminate limit to which we may apply l'Hôpital's Rule. Another indeterminate case is  $\frac{\infty}{\infty}$ . This happens with  $\lim_{x \rightarrow \infty} \frac{\ln x}{x}$ . Both numerator and denominator approach infinity as  $x$  approaches infinity. But upon taking derivatives, we have

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1/x}{1} = \lim_{x \rightarrow \infty} \frac{1}{x} = 0.$$

Using l'Hôpital's Rule, evaluate  $\lim_{x \rightarrow \infty} \frac{x^2}{2^x}$  and  $\lim_{x \rightarrow 0^+} x \ln x$ .

## Lab 8: Sequences

### Goals

- To begin thinking about the qualitative behavior of sequences
- To gain familiarity with important but non-obvious limits
- To develop a feel for various rates of growth of sequences that diverge to infinity

### Before the Lab

Let  $\{a_n\}$  be a sequence of real numbers. For the purposes of this lab, we distinguish four cases.

- I) If  $\lim_{n \rightarrow \infty} a_n = \infty$ , we say  $\{a_n\}$  *diverges to infinity*;
- II) if  $\lim_{n \rightarrow \infty} a_n = -\infty$ , we say  $\{a_n\}$  *diverges to negative infinity*;
- III) if  $\lim_{n \rightarrow \infty} a_n = c$ , a finite real number, we say  $\{a_n\}$  *converges to  $c$* ; and,
- IV) in all other cases, we say that  $\{a_n\}$  *diverges by oscillation*.

In this lab, you will gain insight into the behavior of  $\{a_n\}$  by looking at the way the ordered pairs  $(n, a_n)$  lie when plotted in the plane.

1. Before you begin the lab, decide which of the four types of behavior applies to each of the following sequences.

$$(a) a_n = \frac{n}{\sqrt{n}}$$

$$(b) a_n = \frac{n + (-1)^n}{n}$$

$$(c) a_n = \frac{2^n + n^3}{3^n + n^2}$$

$$(d) a_n = \frac{(-1)^n(n-1)}{n}$$

Now would be a good time to read and do problems 1 and 2 in *TI Lab 9: Sequences and Series* on page 293, if you have not done so already.

### In the Lab

2. Use your graphing calculator to plot the first 50 terms of each sequence in problem 1. Remember to change your MODE to SEQ, and to adjust the WINDOW. Use the same graphical analysis to investigate the behavior of each of the following sequences. Be sure to keep track of your results.

$$(a) a_n = \frac{\ln n}{\sqrt{n}}$$

$$(b) a_n = \frac{n^{10}}{2^n}$$

- (c) Experiment with several other values of  $k$  for sequences of the form  $a_n = \frac{n^k}{2^n}$ . Does changing the value of  $k$  seem to affect the value of the limit? As you change the value of the exponent  $k$  you may need to make changes in your WINDOW in order to see all the points of the sequence.
3. In problem 2(c) you experimented with modifying the exponent. In this problem you will modify the base.
- (a) Write the first few terms of the sequence  $\{r^n\}$  for  $r = 1$  and  $r = -1$ . Which does not converge?
- (b) Determine the behavior of  $\{r^n\}$  for other values of  $r$ , both positive and negative, keeping track of the behavior that you observe: divergent to infinity or negative infinity, convergent to a finite real number (which number?), or divergent by oscillation.
- (c) Tell the whole story. In your written report for this lab, give a complete description of all possible types of behavior for the sequence  $\{r^n\}$  as the value of  $r$  ranges over all the real numbers. State clearly—with specific examples—which values of  $r$  give rise to which type of behavior.
4. Most of the problems in this lab involve the relative rates at which sequences of positive terms diverge to infinity. We clarify the notion of “relative rates” in the following formal definitions.

Let  $\{a_n\}$  and  $\{b_n\}$  be sequences of positive real numbers that diverge to infinity. We say that  $\{a_n\}$  is *strictly faster* than  $\{b_n\}$  if and only if  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$ , and that  $\{a_n\}$  is *strictly slower* than  $\{b_n\}$  if and only if  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$ . If  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$  is nonzero and finite, we say that  $\{a_n\}$  is *comparable* to  $\{b_n\}$ .

On the basis of your earlier work in this lab, which is strictly faster,  $\{n\}$  or  $\{\ln n\}$ ? Are  $\{2^n\}$  and  $\{n^2\}$  comparable? What about  $\{2n - 3\}$  and  $\{3n - 2\}$ ? Explain!

5. In this problem, you will slow down the divergence to infinity of a sequence in two different ways: first by taking the logarithm and then by taking the  $n$ th root of each term.
- (a)  $\{\ln a_n\}$ : The sequence  $\{2^n\}$  is strictly faster than  $\{n\}$ , but taking the logarithm of each of its terms results in a sequence that is comparable to  $\{n\}$ . (Why?) Find another sequence that is strictly faster than  $\{n\}$  but which, when slowed down in this way, becomes strictly slower than  $\{n\}$ . Assemble graphical evidence to support your example.
- (b)  $\{a_n^{1/n}\}$ : Taking the  $n$ th root of the  $n$ th term in the sequence  $\{2^n\}$  results in the constant sequence  $\{2\}$ . Discuss what happens to the sequence  $\{n\}$  when you slow it down in this way. Does  $\{n^{1/n}\}$  have a finite limit? What is it? Give graphical evidence to support your claim.
6. (a) It is a well-known and important fact that  $\{(1 + \frac{r}{n})^n\}$  converges to  $e^r$ . By letting  $r = 1, \ln 2$ , and  $\ln 3$ , assemble evidence in support of this fact.
- (b) Investigate the behavior of two related sequences  $\{(1 + \frac{1}{n^2})^n\}$  and  $\{(1 + \frac{1}{\sqrt{n}})^n\}$ . Discuss how the sequences differ in their behavior as  $n$  gets large. What do you think accounts for this difference?

## Lab 9: Approximating Functions by Polynomials

### Goals

- To introduce the idea of one function being a good approximation to another
- To prepare the student for work on Taylor polynomials and Taylor series

### In the Lab

Polynomials can be easily evaluated at any point and their integrals are easy to find. This is not true of many other functions. Thus, it is useful to find polynomials that are good approximations to other functions.

In this lab we will find polynomials that approximate the exponential function. This function is important in mathematics and frequently appears in models of natural phenomena (population growth and radioactive decay, for instance). In these situations, we need an easy way to approximate  $e^x$  for all values of  $x$ , not just for integers and simple fractions. Also, integrals involving the exponential function are important in statistics. For example, the integral

$$\frac{1}{\sqrt{2\pi}} \int_0^{1/2} e^{-x^2} dx,$$

which calculates the probability of a certain event that follows the “bell-shaped curve” of the normal distribution, simply cannot be evaluated in terms of the usual functions of calculus.

We will rely on the calculator’s ability to evaluate and graph the exponential function in order to determine polynomials that appear to be good approximations to this function. We will use our polynomial approximations to compute integrals involving the exponential function.

1. We begin with a constant function that best approximates  $e^x$  near  $x = 0$ . Why is the graph of  $y = 1$  the best constant approximation to the graph of  $y = e^x$  near  $x = 0$ ? That is, why would  $y = 2$  or  $y = -1$  be a worse approximation to  $y = e^x$  near  $x = 0$ ? Let us denote this polynomial approximation of degree zero by  $P_0$ .
2. Now we want to add a first degree term to  $P_0$  to find  $P_1$ , a polynomial of the form  $1 + ax$  that best approximates  $e^x$  near 0. Use your calculator to graph  $y = e^x$  and several candidates such as  $y = 1 + 0.5x$ ,  $y = 1 + 0.9x$ , and  $y = 1 + 1.2x$  on the same axes. Keep in mind that you are looking for the value of  $a$  so that  $1 + ax$  best approximates  $e^x$  near 0. Thus you should favor a line that follows along the curve  $y = e^x$  right at 0. You may need to change your window to decide which line is better.
3. Next, find the second degree term  $bx^2$  to add to  $P_1$  to get a quadratic polynomial  $P_2(x) = 1 + ax + bx^2$  that best approximates  $e^x$  near 0. Try to get a parabola that follows along the graph of  $y = e^x$  as closely as possible on both sides of 0. Again, record the polynomials you tried and why you finally chose the one you did.
4. Finally, find a third degree term  $cx^3$  to add to  $P_2$  to get a cubic polynomial  $P_3(x) = 1 + ax + bx^2 + cx^3$  that best approximates  $e^x$  near 0. This may not be so easy; you may have to change the window several times before you see why one polynomial is better than another.

5. Now that you have a polynomial that approximates  $e^x$ , try evaluating  $P_3(0.5)$  as a computationally simple way of estimating  $e^{0.5}$ . How close is the polynomial approximation to the value of  $e^{0.5}$  as determined by a calculator? Which is larger? How does the error at the other points of the interval  $[0, 0.5]$  compare with the error at  $x = 0.5$ ? If you cannot distinguish between the graphs of  $y = P_3(x)$  and  $y = e^x$ , you may want to plot the difference  $y = e^x - P_3(x)$  with a greatly magnified scale on the  $y$ -axis.
6. Let us return to the problem of computing a definite integral such as

$$\int_0^{0.5} e^{-x^2} dx$$

for which the integrand does not have an antiderivative in terms of elementary functions. Since  $P_3(x)$  approximates  $e^x$ , we can use  $P_3(-x^2)$  to approximate  $e^{-x^2}$ .

- (a) Evaluate  $\int_0^{0.5} P_3(-x^2) dx$  as an approximation to the above integral.
- (b) Use the numerical integration function on your calculator to approximate the above integral. How does this compare with your answer for part (a)?
7. An analytical method for approximating a function near a point leads to what are known as *Taylor polynomials*. The Taylor polynomial of degree  $n$  is determined by matching the values of the polynomial and its first  $n$  derivatives with those of the function at a particular point.
- (a) Make a table to compare the values of  $P_3$  and its first three derivatives with the values of  $e^x$  and its derivatives, all evaluated at  $x = 0$ . How close was your polynomial  $P_3$  to being a Taylor polynomial?
- (b) Determine the cubic Taylor polynomial for the exponential function. To do this, adjust the four coefficients so the values of the Taylor polynomial and its first three derivatives match those of  $e^x$  at  $x = 0$ . Plot this polynomial and your polynomial  $P_3$ . Compare how close they are to the graph of  $y = e^x$  near 0.

### After the Lab

8. Determine the constant polynomial, the first degree polynomial and the quadratic polynomial that you feel best approximates  $e^x$  on the entire interval  $[-1, 1]$ . Record your attempts. Do all three polynomials have the same constant term? Did you change the coefficient of  $x$  when you went from the straight line to the parabola? What criteria are you using to decide if one polynomial is a better approximation than another?

There are many different methods for approximating a function over a given interval. Do not worry if your answer is quite different from your partner's answer. The point of the problem is to get you thinking about the type of criteria you used to determine the quality of the approximation. If this type of problem appeals to you, you may want to take a course in college called "Numerical Analysis" for further information about the surprising variety of techniques for approximating functions.

## Lab 10: Newton's Method

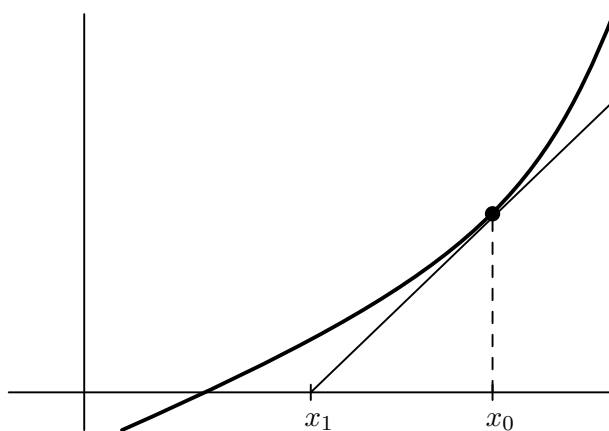
### Goals

- To learn how to use Newton's Method to solve equations
- To understand the geometry of Newton's Method
- To see the importance of the initial guess

### Before the Lab

Very few equations  $f(x) = 0$  can be solved exactly. You have learned methods and tricks for solving equations such as  $x^4 - 5x^2 + 6 = 0$  (factoring),  $x^2 - 6x + 6 = 0$  (quadratic formula), and  $\cos^2 x = \sin x$  (trigonometric identities). However, no general techniques exist for most equations, and we must settle for approximate solutions. The method we will study in this lab is attributed to Isaac Newton and uses the idea that the tangent line to a curve closely approximates the curve near the point of tangency.

Suppose we have a function  $f$ , and we want to solve  $f(x) = 0$ . To use Newton's Method, you must have an initial guess at the solution; we will call this  $x_0$ . The next guess,  $x_1$ , is found at the intersection of the  $x$ -axis with the tangent line to  $y = f(x)$  at  $(x_0, f(x_0))$ . See the figure below.



1. We need to find a formula for  $x_1$ .

- Use  $\frac{\Delta y}{\Delta x}$  to find  $f'(x_0)$ , the slope of the tangent line to  $y = f(x)$  at  $x_0$ , in terms of  $x_0$ ,  $x_1$  and  $f(x_0)$ .
- Solve for  $x_1$  to get the first iteration of Newton's Method:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Once we have  $x_1$ , we repeat the process to get  $x_2$  from  $x_1$ ,  $x_3$  from  $x_2$ , etc. If all goes well, the  $x_i$ 's get closer and closer to the zero of  $f$  that we are seeking.

- Write a formula for  $x_2$  in terms of  $x_1$ .
- Write a formula for  $x_{n+1}$  in terms of  $x_n$ .

**In the Lab**

Computing Newton's Method by hand is tedious and error prone. Since there is a formula for each successive approximation for the solution of  $f(x) = 0$ , the procedure is easily performed by a calculator. The following are instructions for performing Newton's Method in a TI-83 or TI-84 graphing calculator.

- i. Enter the function for which you wish to apply Newton's Method as **Y1**.
- ii. Go back to the home screen and store your initial guess as **X**. (If your initial guess is 3 for instance, press **3 STO X**.)
- iii. Then type **X-Y1/nDeriv(Y1,X,X)** and store this as **X**. (Notice there are no parentheses around **X-Y1**.)

Pressing enter will produce  $x_1$ ; pressing enter again will give  $x_2$ , and so on. Pressing enter repeatedly will produce successive approximations; pressing enter repeatedly until the values do not change gives the best possible approximation with this method.

2. The first application of Newton's Method is to solve the equation  $x^3 - 4x^2 - 1 = 0$ .
  - (a) Graph the function  $f(x) = x^3 - 4x^2 - 1 = 0$  on your paper and notice that the above equation has only one solution.
  - (b) Use Newton's Method with  $x_0 = 5$ . What solution did you find?
3. You can use Newton's Method to find square roots of numbers. For example, to find  $\sqrt{n}$ , solve the equation  $f(x) = 0$  where  $f(x) = x^2 - n$ .
  - (a) Find  $\sqrt{15}$  using Newton's Method, specifying what value you used for  $x_0$ .
  - (b) Check your answer with your calculator. Many calculators, including the TI graphing calculators, use Newton's Method with an initial guess of 1 to take square roots.

The main difficulty in using Newton's Method occurs in the choice of the initial guess,  $x_0$ . A poor choice can lead to a sequence  $x_1, x_2, \dots$  that does not get at all close to the solution you are seeking.

4. Go back to the equation from problem 2.
  - (a) Let  $x_0 = 2$  and use Newton's Method. What seems to be happening? Sketch the first three iterations of Newton's Method on a graph of  $f$ .
  - (b) Let  $x_0 = 0$  and use Newton's Method again. What happens here?
5. In this problem we are going to find the point on the curve  $y = 1/x$  that is closest to the point  $(1, 0)$ .
  - (a) Find the function (in the single variable  $x$ ) that gives the distance from any point on the curve  $y = 1/x$  to the point  $(1, 0)$ .  
(Recall the distance formula is  $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ .)
  - (b) Compute the derivative of the distance function that you found in part (a).
  - (c) Display the necessary work to show that finding the critical points of the distance function leads to solving the equation  $x^4 - x^3 - 1 = 0$ .
  - (d) Now use Newton's Method to find all of the solutions of the equation  $x^4 - x^3 - 1 = 0$ .
  - (e) What point on the curve  $y = 1/x$  is closest to the point  $(1, 0)$ ?





## CHAPTER 12

---

# TI-CALCULATOR LABS

---

## Before You Start the Graphing Calculator Labs...

Class time is limited. There is simply not enough time to go over calculator functions that help you get the most out of your TI-83. These short labs are designed to help you use the built-in functions on your TI-83 (or TI-84), as well as some calculus-related programs. In order to make these labs as meaningful (and as short!) as possible, I will assume that you already know how to do the following on your calculator.

- Change a decimal to fraction;
- Graph a function;
- Find the value of a function;
- Find the zeros (roots) of a function;
- Use the table (including TblSet);
- Set the window;
- Use Zoom In, Zoom Out, ZSquare, ZStandard, and ZTrig;
- Find the intersection point of two functions;
- Enter a list of data and use StatPlot to plot the data; and
- Delete and archive programs.

**If you do not know how to do any of the above, it is vitally important that you see me or a classmate to learn how!**

Finally, here are a few calculator tips:

- Using the TRACE feature is not accurate, particularly for finding zeros or intersection points. Break the habit of using TRACE to find values!
- The answers given by the calculator are not always the right ones! Your calculator is, in fact, a high-speed approximation machine—thus, the value of 1.999999783 that your calculator gives you may actually be 2!
- Radians, Radians, Radians!! (We will never use degree mode, unless specified by the problem.)
- When in doubt, use parentheses. For example, to enter the function  $y = \frac{3}{5x^{12}}$  you must type Y1=3/(5X^(12)) on the Y= screen. Note that Y1=3/5X^12 is not what you want, and will be interpreted by the calculator as  $\frac{3}{5}(x^1)(2)$ .
- You will receive calculus programs that will take up approximately 12,000 bytes of your calculator's program storage. You may need to delete or archive some of your other programs to make room.
- Batteries are your responsibility, as is bringing your own calculator. You are welcome to bring a spare set of batteries (with your name on them) to keep in the classroom. Donations of batteries for someone else to use are welcome also!

### TI Lab 1: Useful Stuff

IN THIS LAB, YOU WILL: EVALUATE FUNCTIONS FROM THE HOME SCREEN;  
 ENTER LISTS FROM THE HOME SCREEN;  
 CHANGE THE GRAPHING STYLE;  
 DISPLAY A GRAPH AND ITS TABLE SIDE-BY-SIDE;  
 SET UP THE TABLE USING VALUES YOU DETERMINE.

1. Enter  $Y1=X^3-4X$  and graph this on the standard “10 by 10” window. To find the value of  $y$  when  $x = 1$ , you can do one of three things:

- From the CALC menu, choose value, press 1, then press ENTER;
- Press TRACE, press 1, then press ENTER;
- Go back to the home screen, and enter  $Y1(1)$ .

(To type  $Y1$  on the home screen, you must choose this from the Variable menu: press VARS, go to the Y-VARS menu, choose 1:Function... and  $Y1$  is the first option. Notice all other  $Y$  variables are listed there.)

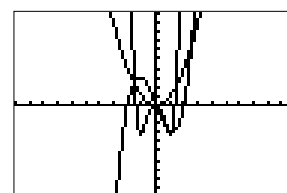
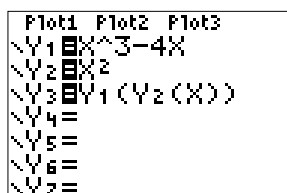
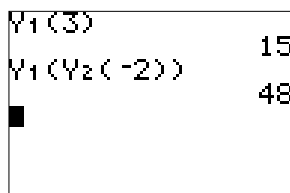
a) Using the home screen entry method, find  $Y1(3)$ .

This method has its advantages. Enter  $Y2=X^2$ . Now you can evaluate the composition of two (or more!) functions.

b) Evaluate  $Y1(Y2(-2))$  and  $Y2(Y1(-2))$ .

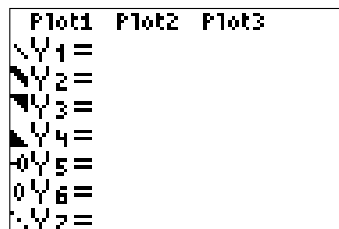
You can also graph the composition of functions.

c) Enter  $Y3=Y1(Y2(X))$  and graph it, along with  $Y1$  and  $Y2$ .



2. All three functions from Problem 1 part (c) are hard to make out when they are graphed on the same screen. Luckily, your calculator allows varying graph styles so you can determine which curve is which. They are

- Normal style*
- Bold style**
- Shade above the graph*
- Shade below the graph*
- Trace and leave a trail*
- Trace but don't leave a trail*
- Dotted*



To change from one style to the other, position the cursor over the current style and press enter repeatedly until the style you want appears.

Using the functions from Problem 1 part (c), change  $Y2$  to dot style and  $Y3$  to bold style. Graph them again.

3. Like evaluating functions, lists do not have to be entered through the STAT EDIT menu—they can be entered from the home screen. To enter the numbers  $-3$ ,  $1$ , and  $2$  into a list, store it as List 1 by typing  $\{-3,1,2\} \rightarrow L1$ . The  $\rightarrow$  symbol (which means “store into”) is made by pressing the STO key. An advantage of this is that functions can be evaluated using lists. To evaluate  $Y1$  at three points  $x = -3$ ,  $x = 1$ , and  $x = 2$ , simply enter the  $x$ -values in  $L1$  and then enter  $Y1(L1)$ . You may also enter the list directly, without storing it by typing  $Y1(\{-3,1,2\})$ .

You may also perform normal arithmetic operations on lists. Evaluate the function in  $Y3$  at the points  $-2, 0, 3$ , and  $5$  using a list.

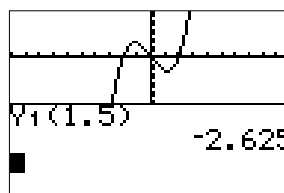
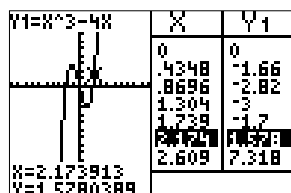
```
{-3,1,2}→L1
{-3 1 2}
Y1(L1)
{-15 -3 0}
Y1({-3,1,2})
{-15 -3 0}
```

```
{-5,2,13}→L1
{-5 2 13}
L1^2
{25 4 169}
-4L1
{20 -8 -52}
```

4. Clear  $Y2$  and  $Y3$ . Sometimes it is convenient to view both a graph and its table. To view both  $Y1$  and its table, press the MODE button and move to the last row. There are three options: Full, Horiz, G-T. (Full should already be highlighted.) Move the cursor to G-T and press ENTER to highlight it. Then press GRAPH. You should see the screen split vertically, with the graph on the left and the table on the right.

Pressing TRACE matches the table with the values from the graph. Like any other graph or table, you may change the window, set up the table, or zoom just as before.

To view the graph and still perform operations on the home screen, change the mode to Horiz. Then the screen is split horizontally, with the graph on the top and the home screen on the bottom.



5. Finally, if you want specific values in your table, you may enter them manually, and the table will show only those values! This is done by selecting Ask on the Indpnt line in the TBLSET menu. Once Ask is selected, pressing TABLE allows you to enter whatever  $x$ -values you want to fill your table.

```
TABLE SETUP
TblStart=0
ΔTbl=1
Indpnt: Auto Ask
Depend: TIME Ask
```

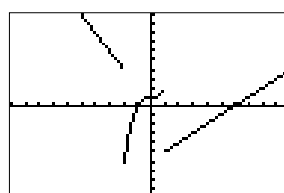
X	Y1	
2	0	
-3.5	-28.88	
12	1680	
6	192	

X=

6. To graph a piecewise function like  $f(x) = \begin{cases} -2x & x < -2 \\ x^3 + 1 & -2 \leq x < 1 \\ x - 6 & x \geq 1 \end{cases}$ , you must enter the each piece

divided by each condition on a separate line. The greater and less than symbols are located in the TEST menu (2nd MATH). The figures below show the function and its graph.

```
Plot1 Plot2 Plot3
Y1= (-2X)/(X<-2)
Y2=(X^3+1)/(X<=
-2)(X<1)
Y3=(X-6)/(X≥1)
Y4=
Y5=
```



## TI Lab 2: Derivatives

IN THIS LAB, YOU WILL: CALCULATE THE DERIVATIVE OF A FUNCTION AT A POINT FROM THE HOME SCREEN;  
 CALCULATE THE DERIVATIVE OF A FUNCTION AT A POINT FROM THE GRAPH;  
 GRAPH THE DERIVATIVE OF A FUNCTION;  
 GRAPH TANGENT LINES; AND  
 USE A PROGRAM TO FIND THE DERIVATIVE OF A FUNCTION DEFINED BY A TABLE OF VALUES.

1. The TI-83 calculates an approximation to the derivative of a function  $f(x)$  around the point  $x = a$  like this:

$$f'(x) \approx \frac{1}{2} \left[ \frac{f(a+h) - f(a)}{h} + \frac{f(a) - f(a-h)}{h} \right].$$

This is basically the definition of the derivative, calculated from both left and right sides of  $a$ , and then the average is taken. The command for the derivative is `nDeriv` and is under the `MATH` menu as choice 8. The format is `nDeriv(function, variable, point, h)`. The calculator automatically uses the value  $h = 0.001$ , but you can specify another value (the smaller the value, the more accurate the answer and the longer it takes to calculate).

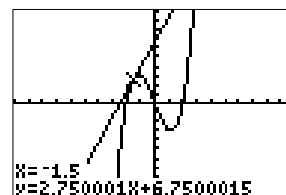
- a) Evaluate the derivative of  $f(x) = x^3 - 4x$  at the point  $x = 4$  by entering `nDeriv(X^3-4X,X,4)`.  
 WARNING: `nDeriv` will not give good results if you mistakenly attempt to evaluate the derivative of a function where the derivative is not defined! For instance,  $f(x) = |x - 2|$  is not differentiable at  $x = 2$ , and so the derivative there is undefined.
- b) Enter `nDeriv(abs(X-2),X,2)`. What did you get?

2. You can use `nDeriv` to graph the derivative of a function without finding the expression for the derivative.

- a) Enter  $X^3 - 4X$  into `Y1`. Enter `Y2=nDeriv(Y1,X,X)`. Then choose `ZStandard`. Obviously the derivative of `Y1` is  $3x^2 - 4$ . Enter this on `Y3` and compare with `Y2` using the table. How accurate is the derivative approximation?
- b) Clear `Y3`. Enter `Y1=sin(X)`, and change the graph style to bold on `Y2`. Make sure the mode is radians and then select `ZTrig`. What function does the derivative resemble?

```
nDeriv(Y1,X,3)
23.000001
nDeriv(Y1,X,3,.0
00001)
23
nDeriv(abs(X-2),
X,2)
```

```
Plot1 Plot2 Plot3
Y1 X^3-4X
Y2 nDeriv(Y1,X,
X)
Y3 =
Y4 =
Y5 =
Y6 =
```



3. The TI-83 can also graph the tangent line to a function at a point. That function is in the `DRAW` menu (`2nd PRGM`).

Clear `Y2`. Enter `Y1=X^3-4X` and choose `ZStandard`. With the graph on screen, choose `tangent` from the `DRAW` menu. Move the cursor to a point at which you want the tangent (or enter the  $x$ -value) and press enter. Notice that the equation of the tangent is at the bottom of the screen. What is the approximation to the tangent line? (When you are finished, you must choose `ClrDraw` from the `DRAW` menu to remove the tangent line, or clear the `Y=` screen.)

4. There is also a way to calculate the derivative directly on the graph by using the  $dy/dx$  function on the **CALC** menu (it is choice 6). Move the cursor to the point at which you want to calculate the derivative (or enter the value) and press **ENTER**. Remember that this is only an approximation.

Graph  $y = x^3 + x^2 + e^{-x/2}$  on the standard window and find the derivative at the points  $x = 1$ ,  $x = 2$ , and  $x = -0.95$ . What conclusion can you draw from the value of the derivative at  $x = -0.95$ ?

5. Finally, the program **DERDATA** calculates the approximate derivative from a function defined by a table of values. The  $x$ -values must be entered in **L1** and the  $y$ -values go in **L2**.

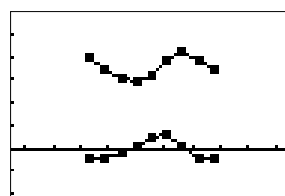
- a) The following table gives the unemployment rate (as a percentage) in the U.S. for the years listed. Estimate the rate of change (the derivative) in the unemployment rate for the years 1987 and 1992.

year ( $x$ )	1986	1987	1988	1989	1990	1991	1992	1993	1994
% ( $y$ )	7.0	6.2	5.5	5.3	5.6	6.8	7.5	6.9	6.1

```

THE DERIVATIVE'S
X- AND Y-VALUES
WILL BE STORED
IN L3 AND L4.
ENTER NUMBER OF
DATA POINTS:
?9

```



- b) Estimate the derivative at  $x = -1$  and at  $x = 2$  for the function defined by the following table.

$x$	-6	-5	-2	-1	0	2	3	7
$y$	-3.8	-2.85	-1.66	-1.5	-1	0.08	0.83	5.83

## TI Lab 3: Maxima, Minima, Inflections

IN THIS LAB, YOU WILL: FIND BETTER APPROXIMATIONS FOR THE MAXIMUM AND MINIMUM OF A FUNCTION; AND  
USE A PROGRAM TO DETERMINE POINTS OF INFLECTION.

1. As you know, the maximum and minimum values of a function can be found by using **max** or **min** from the **CALC** menu. These values, though easy to find, aren't as exact as they could be. The first derivative test gives us another way to calculate extreme values: we graph the derivative and determine its zeros.

Graph  $y = \frac{\sqrt{4+x^2}}{3} + \frac{10-x}{4}$  and its derivative (remember, you do not have to find the derivative in order to graph it). Use the **min** function to find the minimum of  $y$ , then use the **zero** function to find the zero of the derivative. Which is more exact? Use your table on **Indpnt:Ask** to find out.

2. The program **INFLECT** will locate inflection points on a graph. The program uses **nDeriv** twice—it simply looks for zeros of the second derivative. This program needs you to tell it where to start looking for the inflection point. So it asks for left and right bounds, just like **max**, **min**, or **zero** functions.

- Use **INFLECT** to find the inflection point of  $y = x^3 - 2x^2 - 4x + 3$ .
- Find the inflection points of  $y = x^3 + x^2 + x + e^{-x}$ .
- Determine the intervals where  $y = e^{x/2} - \ln(x^3 + 1)$ , for  $x > -1$ , is increasing, decreasing, concave up, and concave down. Use the window  $-1 \leq x \leq 4$  and  $-2 \leq y \leq 3$ .

```
THIS PROGRAM
FINDS INFLECTION
POINTS OF A
FUNCTION.

<PRESS ENTER>
```

```
Y1=X^3+X^2+X+e^(-X)
RIGHT BOUND?
X=-1.914894 Y=1.5165752
```

```
Y1=X^3+X^2+X+e^(-X)
LEFT BOUND?
X=-2.978723 Y=-.8728275
```

```
INFlectPt
X=-2.617963 Y=7.2715E-4
```

## TI Lab 4: Integrals

IN THIS LAB, YOU WILL: CALCULATE DEFINITE INTEGRALS FROM THE HOME SCREEN;  
 CALCULATE DEFINITE INTEGRALS FROM A GRAPH;  
 CALCULATE DEFINITE INTEGRALS OF PIECEWISE FUNCTIONS;  
 GRAPH THE INTEGRAL OF A FUNCTION;  
 DISCOVER A FUNCTION THAT IS DEFINED BY AN INTEGRAL; AND  
 USE A PROGRAM TO FIND THE AVERAGE VALUE OF A FUNCTION.

**1.** Just as you calculate derivatives at a point, you can compute definite integrals on your calculator. The command `fnInt` approximates a definite integral. The format is `fnInt(function, variable, lower, upper, tolerance)`, where tolerance is the degree of accuracy with which you wish to compute the integral. If you do not specify a value, the calculator assumes that the tolerance is 0.0001.

a) Evaluate  $\int_0^1 \frac{4dx}{1+x^2}$ .

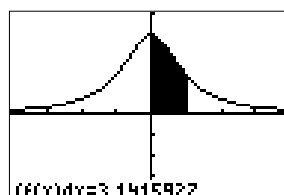
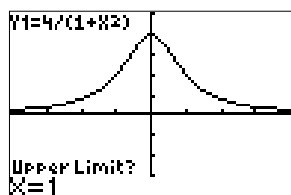
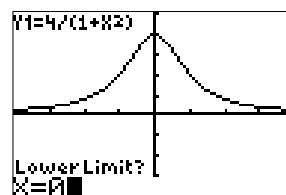
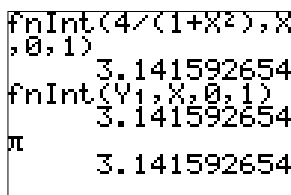
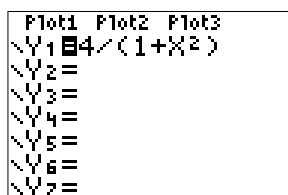
b) Evaluate  $\int_{-1}^3 \frac{\sin x}{x} dx$ . Make sure you are using radians!

(Notice that the calculator evaluates the integral in part (b) even though the integrand is not defined at  $x = 0$ !)

**2.** Like `nDeriv`, you can store the function you wish to integrate as `Y1` and enter `fnInt(Y1,X,lower,upper)`. The advantage to entering the function in `Y1` is that there is also an integral function located under the `CALC` menu—`7: ∫ f(x)dx`. Not only will the `∫ f(x)dx` command evaluate the definite integral, it will shade the area the integral represents!

a) Enter `Y1=sin(X)/X` and graph it using `ZTrig`. Use `∫ f(x)dx` to evaluate  $\int_{-1}^1 \frac{\sin x}{x} dx$ . Enter the lower or upper limits (or `TRACE` them) and press `ENTER`.

b) Graph the function  $f(x) = \begin{cases} x+3 & x \leq 3 \\ -(x-5)^2 & x > 3 \end{cases}$ . Then evaluate  $\int_{-5}^7 f(x) dx$  and  $\int_{2.5}^{3.5} f(x) dx$ .



**3.** You may also graph the integral of a function.

a) Enter `Y1=3X^2-4` and enter `Y2=fnInt(Y1,X,0,X)`. This graphs the integral of `Y1`. Clearly, the integral is equal to  $x^3 - 4x$ . Enter this expression on `Y3` and compare the accuracy of `Y2` using your table.

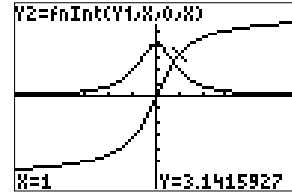
You probably noticed that the graph of `Y2` goes very slow. This is because the calculator is evaluating the integral at every pixel. Adjusting `Xres` in the `WINDOW` menu to a higher value (say 2 or 3) will reduce graphing time, but will also reduce accuracy.



```

Plot1 Plot2 Plot3
Y1=4/(1+X^2)
Y2=fnInt(Y1,X,0,X)
Y3=
Y4=
Y5=
Y6=

```



- b) Find the position of a particle at  $x = -\frac{1}{2}$  and  $x = 1$  if the particle's velocity is  $\frac{1}{\sqrt{4-x^2}}$ , for  $-2 < x < 2$ .
- c) Clear your Y= screen. Enter and graph  $\text{nDeriv}(\text{fnInt}(X^3-4X, X, 0, X), X, X)$ . Change Xres to 2 and use ZStandard. (This will take up to 40 seconds to graph.) Does the graph look familiar? Which function is it?
4. Many important functions are defined as integrals. Consider the function

$$L(x) = \int_1^x \frac{1}{t} dt,$$

for  $x > 0$ . The function  $L$  is undefined for  $x > 0$  since  $f(t) = 1/t$  is not continuous at  $t = 0$ .

- a) Graph  $Y1=\text{fnInt}(1/T, T, 1, X)$  in the window  $0.01 \leq x \leq 10$  and  $-1.5 \leq y \leq 2.5$ .
- b) For what values of  $x$  is  $L(x) = 0$ ?
- c) Use the table to make a list of approximate values for  $L(x)$  using 8 equally spaced values starting with  $x = 1$ .
- d) Verify from the table created in part (c) that:
- $$L(6) = L(2) + L(3), \quad L(4) = 2L(2), \quad L(8) = 3L(2)$$
- e) From the home screen, enter  $\{1, 2, 3, 4, 5, 6, 7, 8\} \rightarrow L1$  then enter  $Y1(\text{Ans}) \rightarrow L2$ . This puts the  $x$ -values and  $y$ -values from the table in part (c) into lists. Now, from the STAT CALC menu, choose LnReg to fit a logarithmic regression curve to the data. What is the equation of the curve that best fits the data?
- f) What is the derivative of  $L(x)$ ?

## TI Lab 5: Approximating Integrals with Sums

IN THIS LAB, YOU WILL: APPROXIMATE A DEFINITE INTEGRAL USING A RIEMANN SUM PROGRAM.

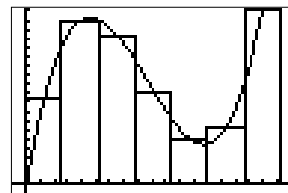
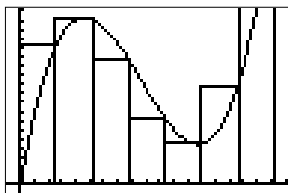
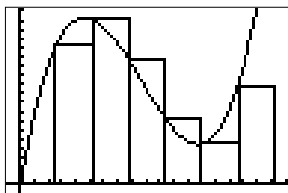
1. Now that you have learned how to approximate a definite integral using a Riemann Sum, it's time you learned how to do this on your calculator by using the program RIEMANN. The first screen you see upon running the program is the one below on the left. Not only can you calculate left- and right-hand sums, you can also calculate midpoint sums and sums of trapezoids that approximate the area. To use this program, you **must always** choose option 1:SET PARAMETERS. This is where you enter the function and set upper and lower bounds.

- a) Run the program. Enter  $x^3 - 10x^2 + 26x$ , lower bound 0, upper bound 7, and 7 partitions.

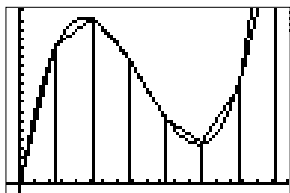
```
RIEMANN SUM
1:SET PARAMETERS
2:LEFT SUM
3:RIGHT SUM
4:MIDPOINT SUM
5:TRAPEZOID SUM
6:DEF. INTEGRAL
7:QUIT
```

```
FUNCTION:
X^3-10X^2+26X
LOWER BOUND:0
UPPER BOUND:7
PARTITIONS:7
```

- b) When you select one of the sums, the program graphs the function over the interval  $[lower, upper]$  and draws the appropriate rectangles. Find the left-hand sum, right-hand sum, and midpoint approximations for the function in part (a).



- c) Next, select the trapezoid approximation. You will learn more about approximating a definite integral with trapezoids in the next Calculus Lab. Finally, select the definite integral. This option is here so you can immediately compare the more exact value with the various approximations.



```
DEF. INTEGRAL:
93.91666667
```

2. Find the left-hand, right-hand, midpoint, and trapezoid sum approximations for the function

$$e^{-x} \ln(x+2) - x + x^2 - \frac{1}{5}x^3$$

over the interval  $[0, 6]$ . Use 6 partitions.

3. Repeat problem 2 with 12 partitions.

## TI Lab 6: Approximating Integrals with Sums II

IN THIS LAB, YOU WILL: USE PROGRAMS TO APPROXIMATE A DEFINITE INTEGRAL IN VARIOUS WAYS

1. The program RIEMANN allows us to estimate various rectangular approximations and a trapezoid approximation to a definite integral, if we know what the function is. Often, we are simply given a table of data and asked to estimate the value of the definite integral of the function represented by the table. The RIEMANN program does not allow the function to be defined by a table. Luckily, you have another program that does just that, using trapezoids: TRAPDATA.

```
MAKE SURE THE
X-VALUES ARE IN
L1 AND THE Y-
VALUES ARE
IN L2.

<PRESS ENTER>
```

```
ENTER NUMBER OF
DATA POINTS:
?
```

To use TRAPDATA, you must first enter the  $x$ - and  $y$ -values in L1 and L2, respectively. Then the program prompts you for the number of data points and returns the trapezoid rule approximation.

- a) In an experiment, oxygen was produced at a continuous rate. The rate of oxygen produced was measured each minute and the results are given in the table below. Use TRAPDATA to estimate the total amount of oxygen produced in 6 minutes.

minutes	0	1	2	3	4	5	6
oxygen (ft <sup>3</sup> /min)	0	1.4	1.8	2.2	3.0	4.2	3.6

- b) All the information given about the continuous function  $f$  is found in the table below. Estimate the area of the region below the graph of  $f$  and above the  $x$ -axis over the interval  $[1, 2]$ .

$x$	1	1.2	1.4	1.6	1.8	2
$f(x)$	7.3	6.8	4.9	5.4	6.0	5.8

2. The program SIMPDAT also approximates the value of a definite integral of a function defined by a table. This program, however, uses Simpson's Rule.

```
MAKE SURE THE
Y-VALUES ARE
IN L1 AND THE
X-VALUES ARE
EQUALLY SPACED.

<PRESS ENTER>
```

```
NUMBER OF DATA
POINTS:
?6
WIDTH OF EACH
INCREMENT(dx):
?1
```

SIMPDAT is not as user friendly as TRAPDAT. You can only use this program if the  $x$ -values are equally spaced (in other words, if the change in each  $x$ -value is the same) and there are an even number of sub-intervals. Since the  $x$ -values are equally spaced, there is no need to enter the  $x$ -values—you are prompted for the number of data points and the width of each subinterval (how far apart the  $x$ -values are spaced). Note that the  $y$ -values go in L1.

- a) Repeat problem 1 part (a) using SIMPDAT.
- b) Compare your answers. Which is more accurate: the answer given by the trapezoid rule or by Simpson's Rule?

- c) Repeat problem 1 part (b) using SIMPDAT. Why is the answer such an underestimate compared to using the trapezoid rule?

3. Finally, the program SIMPEQ approximates a definite integral of a function if the equation is known! The only qualification is that there must be an even number of subintervals.

```
THIS PROGRAM
FINDS THE INTE-
GRAL OF A FUNC-
TION FROM X=A
TO X=B USING
SIMPSON'S RULE.

<PRESS ENTER>
```

```
ENTER EQUATION:
Y1=?e^(-X^2)
A:
?0
B:
?π
IncreMents:
?10
```

```
THE INTEGRAL IS
.8862186166
Done
```

- a) Run SIMPEQ and enter the function  $x^3 - 10x^2 + 26x$ . Estimate the definite integral over  $[0, 7]$  using 6 increments.
- b) Repeat part (a) with 12 increments.
- c) Using Simpson's Rule, estimate the value of  $\int_{0.32}^{\pi} \sin(1/x) dx$  to three decimal places.

## TI Lab 7: Applications of Integrals

IN THIS LAB, YOU WILL: USE A PROGRAM TO APPROXIMATE THE ARC LENGTH OF A CURVE; AND  
USE A PROGRAM TO APPROXIMATE THE VOLUME OF A SOLID OF REVOLUTION.

1. The program **ARC** uses the the calculator's definite integral function to determine the arc length of a function. In the figures below, the arc length of  $\sin x$  from 0 to  $\pi$  is calculated. Notice that the program shaded the part of the curve whose length you found.

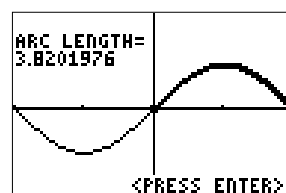
```
THIS PROGRAM
COMPUTES ARC
LENGTH ALONG
F(X) BETWEEN
A AND B

<PRESS ENTER>
```

```
ENTER EQUATION
Y1=?sin(X)

<PRESS ENTER>
```

```
LOWER BOUND=0
UPPER BOUND=π
```



- Set up the integral that represents the length of the curve  $\arctan x$  from 0 to  $\pi$ , then use the program to find the length.
  - Set up the integral that represents the length of the curve  $e^x + e^{-x}$  from  $-1$  to  $1$ , then use the program to find the length.
2. The program **VOLUME** determines the volume of a solid of revolution. This program does a lot to accurately compute the volume and draw the solid—therefore, it is quite involved!
- In this example, we compute the volume of the solid formed by revolving the region between  $x^2$  and the  $x$ -axis from 1 to 3 around the line  $y = -1$ .

We begin with the Area Menu. This is the first screen you see upon running the program. You must choose options 1 through 4 (in that order) before you choose option 6 to compute volume!

```
AREA MENU
1: CURVES
2: WINDOW
3: LOWER BOUND
4: UPPER BOUND
5: SHADE AREA
6: VOLUME MENU
7: QUIT
```

After selecting option 1, you are asked to choose which type of curve you have. If the solid you wish to form is revolved around a horizontal axis, choose  $F(X)$ ; if it is revolved around a vertical axis, choose  $F(Y)$ . In this case, we have a horizontal axis of rotation.

Then you are asked to enter the curves. (Notice that it says “in terms of  $t$ .” This is because the calculator will graph the curve parametrically in order to then graph the solid formed.) Here, we enter  $t^2$  as the first curve and 0 as the second, since the second curve is the  $x$ -axis.

The next screen asks you to set the graph window, then the graph is shown. Notice that you want to set the window appropriately so that you will be able to see the entire region once it is rotated.

```

CURVE TYPE:
 1=F(X)
 2=F(Y)
SELECTION=1

```

```

ENTER CURVE 1
IN TERMS OF T:
CURVE 1=T2

ENTER CURVE 2
IN TERMS OF T:
CURVE 2=0

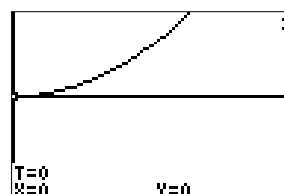
```

```

ENTER CURVE 1
IN TERMS OF T:
CURVE 1=T2

ENTER CURVE 2
IN TERMS OF T:
CURVE 2=0

```



Next, you must set the upper and lower bounds for the region. For both bounds, you are given the option of entering the bound (manual) or tracing the graph to select the bound. Since we know exactly what are bounds are, we choose Manual for both. Finally, you may choose option 5 to shade the region you will be rotating. After the shading is complete, pressing ENTER gives you the area of the region.

```

CHOOSE METHOD:
 1=MANUAL
 2=GRAPHICAL
SELECTION=1

```

```

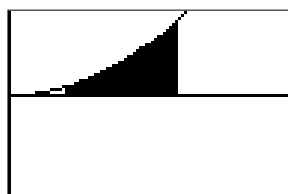
LOWER BOUND=1

```

```

UPPER BOUND=3

```



```

LOW =1.0000
UP  =3.0000
AREA=8.6667

```

Finally, we are ready for the Volume Menu. This is where you choose the rotation axis. You may also choose to have the calculator graph a representative rectangular region in the solid to revolve, although this is not necessary.

```

VOLUME MENU
1:AXIS ROTATION
2:DRAW RECTANGLE
3:ROTATE RECTANG
4:CLEAR DRAW
5:ANSWER
6:AREA MENU
7:QUIT

```

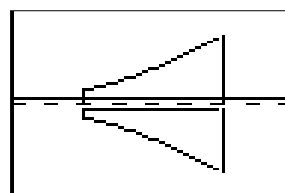
In our example, we want to revolve the region around  $y = -1$ . We choose option 1 and we are asked to choose whether the line is horizontal or vertical and then enter the line. Then the calculator graphs an “outline” of the solid with the axis of rotation as a dotted line. To draw a representative rectangular region, we choose option 2, and we are shown the graph of the region. Move the cursor up to any location on the graph of the function and press ENTER. Then a shaded rectangle is drawn. Next, choose 3 to rotate the rectangle. Finally, choose 5 for the answer to the volume question.

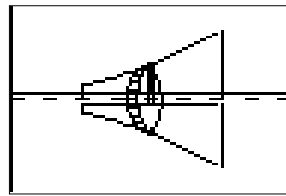
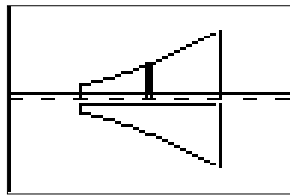
```

CHOOSE TYPE:
 1=HORIZONTAL
 2=VERTICAL
SELECTION=1

HORIZONTAL AXIS:
Y=-1

```





```

LOW BND=1.0000
UP BND =3.0000
VOLUME =206.5074

METHOD IS WASHER
ABOUT THE LINE
Y=-1

```

Drawing the figure is not necessary to compute the answer. Once the axis of rotation is selected, you can go immediately to the answer. Also notice that if you mess up, you have the option of going back to the Area Menu and re-entering your functions and bounds.

- b) Find the volume of the solid formed by revolving the region between  $y = x^2$  and  $y = e^x$  about the  $y$ -axis from  $y = 1$  to  $y = 8$ .

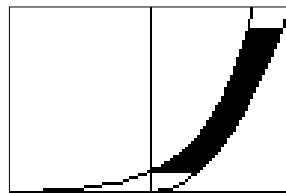
Since the rotation axis is vertical, we must select curve type F(Y). This requires that both functions and both limits must be entered in terms of  $y$ . The limits are already given in terms of  $y$ ; simply solve both functions for  $y$  to get  $x = \sqrt{y}$  and  $x = \ln y$  and you are ready to find the volume.

```

ENTER CURVE 1
IN TERMS OF T:
CURVE 1=√(T)

ENTER CURVE 2
IN TERMS OF T:
CURVE 2=ln(T)

```



```

LOW =1.0000
UP =8.0000
AREA=4.7827

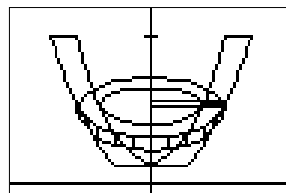
```

```

CHOOSE TYPE:
1=HORIZONTAL
2=VERTICAL
SELECTION=2

VERTICAL AXIS:
X=0

```



```

LOW BND=1.0000
UP BND =8.0000
VOLUME =50.8261

METHOD IS WASHER
ABOUT THE LINE
X=0

```

- c) The region  $R$  is bounded by the curves  $y = \frac{-5(x+2)}{x} + 10$  and  $y = \ln x$  from  $x = 3$  to  $x = 7$ . Set up an integral that represents the volume of the solid generated by revolving the region around the line  $y = 6$ , then find the volume.
- d) The region  $R$  is bounded by the curves  $y = x^3$  and  $y = e^{x/2}$  from  $y = 3$  to  $y = 7$ . Set up an integral that represents the volume of the solid generated by revolving the region around the line  $x = 1$ , then find the volume.

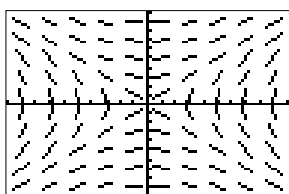
## TI Lab 8: Differential Equations

IN THIS LAB, YOU WILL: USE A PROGRAM TO APPROXIMATE A NUMERICAL SOLUTION TO A DIFFERENTIAL EQUATION USING EULER'S METHOD; AND USE A PROGRAM TO GENERATE A SLOPE FIELD.

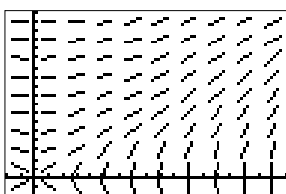
**1.** The program SLOPEFLD draws a slope field for a differential equation. To use this program, the differential equation must be solved for  $y'$ . Enter the resulting equation in Y1 in terms of X and Y (Y is found by pressing ALPHA and 1), then adjust your window to the appropriate size. Then run the program.

The only limitation to this program is that the program always draws 100 slope segments in a 10 by 10 array. Thus, if you want slopes at integer points, you will need a window of  $-0.5 < x < 9.5$  and  $-0.5 < y < 9.5$ . This will create slope segments at integer values along each axis (up to 9) and throughout the first quadrant (up to the point (9,9)). Any multiple of this window will create slope segments at integer points as well: for instance, multiplying by 2 results in a window of  $-1 < x < 19$  and  $-1 < y < 19$  and creates slopes at even integer points. If you want other quadrants as well, simply shift the window: subtracting 9 from both  $x$  and  $y$  results in a window of  $-10 < x < 10$  and  $-10 < y < 10$  and creates slopes at odd integer points.

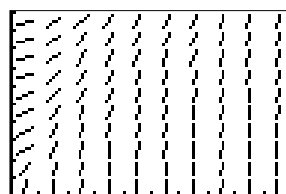
Plot a slope field for the differential equation  $y' = x/y$ . The graphs below show you the difference a window makes.



$-10 < x < 10, -10 < y < 10$



$-2 < x < 18, -2 < y < 18$



$0 < x < 50, 0 < y < 20$

Try using the  $-0.5$  by  $9.5$  window. What happened? Why?

**2.** The program EULER approximates the value of a differential equation and is self-explanatory. The differential equation must be solved for  $y'$ . Enter the resulting equation in Y1 in terms of X and Y, and run the program. "FINAL X" is the point at which you want the approximation. Press enter repeatedly to get the successive approximations.

Plot a slope field for the differential equation  $x + y' = xy$ . Use an appropriate window to get a good idea of what happens around the origin. Then use Euler's method to approximate  $y$  at  $x = 2$  with an initial value  $y(0) = 1$  and step size  $0.2$ .

```

INITIAL
CONDITIONS
X0=0
Y0=1
FINAL X
X=2
STEP SIZE?
H=.2

```



## TI Lab 9: Sequences and Series

IN THIS LAB, YOU WILL: GENERATE A LIST OF TERMS IN A SEQUENCE;  
 FIND THE SUM OF A FINITE NUMBER OF TERMS IN A SEQUENCE;  
 GRAPH A SEQUENCE; AND  
 USE A PROGRAM THAT FINDS THE SUM OF FINITE TERMS  
 OF A GEOMETRIC SERIES.

1. Your calculator can generate terms of a sequence easily and quickly. Go to the LIST menu (2nd STAT) and over to OPS. Choice 5 is the sequence command. The format for the sequence command is  $\text{seq}(\text{sequence}, \text{variable}, \text{start}, \text{end}, \text{increment})$ , where *start* and *end* refer to the term numbers and *increment* tells the calculator how to count from start number to end number. For instance, the first 4 terms of the sequence  $1/n$  are given in the figure below. Note that you can change the sequence to fractions and store the sequence as a list.

```
seq(1/X,X,1,4,1)
(1 .5 .33333333...
seq(1/X,X,1,4,1)
Frac
(1 1/2 1/3 1/4)
```

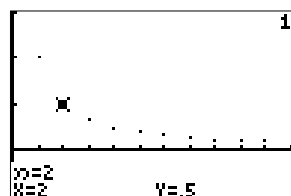
```
seq(1/X,X,1,4,1)
→L1
(1 .5 .33333333...
L1→Frac
(1 1/2 1/3 1/4)
```

2. Your calculator can also graph terms in a sequence. To do this, your calculator's MODE must be Seq and Dot. When you press the Y= button, you get the screen shown below.  $n_{\text{Min}}$  is the starting value of  $n$  and  $u(n)$  is the sequence. The window screen is different as well. In addition to the  $x$  and  $y$  scales, you have  $n_{\text{Min}}$ ,  $n_{\text{Max}}$ , PlotStart and PlotStep. Clearly, the range of  $n$  values you wish to graph is  $n_{\text{Min}}$  and  $n_{\text{Max}}$ . PlotStart is the  $n$  value you wish to begin graphing and PlotStep is the increment. Finally, pressing the graph button produces a graph over which you may trace values. The table is also available in Seq mode as well.

```
Normal Sci Eng
Float 0123456789
Radian Degree
Func Par Pol Seq
Connected Dot
Sequential Simul
Real a+bi re^θi
Full Horiz G-T
```

```
Plot1 Plot2 Plot3
nMin=1
u(n)=1/n
u(nMin)=
v(n)=
v(nMin)=
w(n)=
w(nMin)=
```

```
WINDOW
nMin=1
nMax=10
PlotStart=1
PlotStep=1
Xmin=0
Xmax=11
Xscl=1
```



n	u(n)
1	1
2	.5
3	.33333
4	.25
5	.2
6	.16667
7	.14286

3. To find the sum of some finite amount of terms, go to the LIST menu and over to MATH. Choice 5 is sum. The figure below shows the format to obtain the sum of terms in a sequence.

```
sum(seq(1/X,X,1,
4,1))Frac 25/12
sum(seq((-1)^X/X
,X,1,4,1))Frac -7/12
```

```
sum(seq(3X-1(X),
X,1,40,1))
2288.384212
sum(seq(4(.8)^X,
X,1,50,1))
15.99977164
```

4. The program SERIES combines the summing and graphing features in one program—however, it is only good for geometric sequences. Here, we run the program on the geometric sequence  $\{4(0.8)^n\}$ . The advantage to using this program is that it will give you the value of the infinite sum (if the geometric series converges), and it graphs the partial sums so that you may observe the convergence or divergence of the series.

```
FIRST TERM=A
RATIO=R
NO. OF TERMS=N

A=?4
R=? .8
N=?50■
```

```
PRESS ENTER TO
SEE SUCCESSIVE
TERMS OF SERIES
```

```
16/5
64/25
256/125
1024/625
4096/3125
1.048576
```

```
2.722258935E-4
2.177807148E-4
1.742245719E-4
1.393796575E-4
1.11503728E-4
8.920298079E-5
7.136238464E-5
```



# CHAPTER 13

---

## CHALLENGE PROBLEMS

---

THE FOLLOWING GUIDELINES APPLY FOR ALL THE PROBLEMS IN THIS CHAPTER. PLEASE NOTE THE FOLLOWING.

- NO CREDIT IS GIVEN FOR JUST AN ANSWER. ALL WORK MUST BE SHOWN.
- EACH PROBLEM IS WORTH ANYWHERE FROM 5 TO 25 POINTS, DEPENDING ON THE LEVEL OF DIFFICULTY.
- ALL POINTS EARNED WILL BE ADDED TO YOUR FINAL AVERAGE BEFORE THE SEMESTER EXAM.
- YOUR WORK WILL NOT BE RETURNED TO YOU.
- THE NUMBER OF POINTS AWARDED IS NOT NEGOTIABLE.
- SLOPPY, UNORGANIZED, OR ILLEGIBLE WORK WILL NOT BE ACCEPTED.
- I RESERVE THE RIGHT TO ASK YOU TO EXPLAIN YOUR SOLUTIONS, TO INCLUDE PROBLEMS SIMILAR TO THOSE YOU SOLVE ON YOUR FINAL EXAM, AND TO GIVE YOU NO CREDIT WHATSOEVER IF YOUR SOLUTIONS ARE SIMILAR TO SOMEONE ELSE'S SOLUTIONS.

## Challenge Problem Set A

THESE PROBLEMS MAY BE COMPLETED AND TURNED IN AT ANY TIME BEFORE 3:05 PM, SEPTEMBER 15

- A1.** [SH] Consider the curve  $x^2 - xy + y^2 = 9$ .
- Write an expression in terms of  $x$  and  $y$  for the slope of the curve.
  - Find the coordinates of the points on the curve for which the tangent lines are vertical.
  - At the point  $(0, 3)$ , find the rate of change in the slope of the curve with respect to  $x$ .
- A2.** [RP] Prove that at no point on the graph of  $y = \frac{x^2}{x-1}$  is there a tangent line whose angle of inclination with the  $x$ -axis is  $45^\circ$ .
- A3.** [TICAP] At  $x$  feet above sea level, a person weighing  $P$  pounds at sea level will weigh  $W = \frac{PR^2}{(R+x)^2}$ , where  $R \approx 21,120,000$  feet is the radius of the Earth.
- Let  $P = 110$  pounds be Rachel's weight, and compute the tangent line approximation to  $W$  at  $x_0 = 0$ , the initial height of 0 feet above sea level.
  - Use the tangent line to approximate the weight of Rachel if she is at the top of the 20,320 foot peak of Mount McKinley.
  - Use the formula to perform part (b) exactly.
  - Compare your answers in parts (b) and (c). How accurate is the tangent approximation? Does being at the top of Mount McKinley result in a significant difference in weight?
- A4.** [SS] A function  $f(x)$  is defined as  $f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ x^3 & \text{if } x \text{ is irrational.} \end{cases}$
- Draw the graph of  $f(x)$ .
  - If the limit of  $f(x)$  as  $x \rightarrow 2$  exists, find it.
  - If the limit of  $f(x)$  as  $x \rightarrow 1$  exists, find it.
  - For which real numbers  $a$  does the limit of  $f(x)$  as  $x \rightarrow a$  exist?
  - What is the limit of  $f(x)$  as  $x \rightarrow \infty$ ?
- A5.** [AO] Prove that there is only one value of  $k = 1$  for which the limit, as  $x \rightarrow 0$ , of the function  $\frac{\sin(2 \sin x)}{xk}$  is a real number. Then find this value of  $k$  and evaluate the limit.

## Challenge Problem Set B

THESE PROBLEMS MAY BE COMPLETED AND TURNED IN AT ANY TIME BEFORE 3:05 PM, OCTOBER 27

- B1.** [RP] Given the relation  $x^2y + x - y^2 = 0$ , find the coordinates of all points on its graph where the tangent line is horizontal.
- B2.** [RP] A builder is purchasing a rectangular plot of land with frontage on a road for the purpose of constructing a rectangular warehouse. Its floor area must be 300,000 square feet. Local building codes require that the building be set back 40 feet from the road and that there be empty buffer strips of land 25 feet wide on the sides and 20 feet wide in the back. Find the overall dimensions of the parcel of land and building which will minimize the total area of the land parcel that the builder must purchase.
- B3.** [RP] A driver on a desert road discovers a hole in the gas tank leaking gas at the constant rate of 4 gallons per hour. This driver, having no way to plug the hole, decides to drive for as long as the gas supply allows. The gauge reading indicates the tank is three-fourths full, which means that the tank contains 14 gallons. The car consumes gas at the rate of 18 miles per gallon at 40 mph. For each 5 mph below 40 mph add one-half mile per gallon to this rate; for each 5 mph above 40 mph, subtract one mile per gallon from this rate. If the driver chooses the best constant speed in order to get the maximum driving distance, find the maximum distance that the 14 gallons will allow. Assume that gas consumption is a continuous function of speed.
- B4.** [CG] Analyze the graph of  $y = \cos(2x) - \sin(2x)$  over the interval  $[0, 2\pi)$ . Find the  $x$ -intercepts,  $y$ -intercepts, local maximums, local minimums, intervals where the graph increases and decreases, intervals where the graph is concave up and concave down, and inflection points. Then sketch it as accurately as possible. Show all the analysis that leads to your conclusions; answers may not come from the calculator.
- B5.** [SH] Find the lengths of the sides of the isosceles triangle with perimeter 12 and maximum area.
- B6.** [RP] One ship,  $A$ , is sailing due south at 16 knots and a second ship,  $B$ , initially 32 nautical miles south of  $A$ , is sailing due east at 12 knots.
- At what rate are they approaching or separating at the end of one hour?
  - When do they cease to approach one another and how far apart are they at this time? What is the significance of this distance?

**B7.** [AP76] For a differentiable function  $f$ , let  $f^*$  be the function defined by

$$f^*(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{h}.$$

- (a) Determine  $f^*(x)$  for  $f(x) = x^2 + x$ .
- (b) Determine  $f^*(x)$  for  $f(x) = \cos x$ .
- (c) Write an equation that expresses the relationship between the functions  $f^*(x)$  and  $f'$ , where  $f'$  denotes the usual derivative of  $f$ .

**B8.** [AP76] Consider the function  $f(x) = e^{\sin x}$  for all  $x$  in the interval  $(-\pi, 2\pi)$ .

- (a) Find the  $x$ - and  $y$ -coordinates of all maximum and minimum points on the given interval. Justify your answers.
- (b) Draw the graph of the function. Show the analysis that led to your graph. Do not use your calculator.
- (c) Write the equation for the axis of symmetry of the graph.

**B9.** [RP] Let  $f$  and its first two derivatives be continuous on  $[0, \infty)$  such that

- I.  $f'(0) = 0$
- II.  $f$  is increasing on  $(0, \infty)$
- III.  $\lim_{x \rightarrow \infty} f'(x) = 0$

Prove that  $f$  has at least one inflection point on  $(0, \infty)$ .

## Challenge Problem Set C

THESE PROBLEMS MAY BE COMPLETED AND TURNED IN AT ANY TIME BEFORE 3:05 PM, DECEMBER 8

NO CALCULATOR IS ALLOWED FOR ANY OF THESE PROBLEMS.

- C1.** [RP] Consider the function  $f(x) = x^2 \ln x$  over the interval  $(0, 1)$ .
- Find the coordinates of any points where the graph of  $f(x)$  has a horizontal tangent line.
  - Find the coordinates of all relative and/or absolute maximum or minimum points.
  - Find the coordinates of any points of inflection on the graph of  $f(x)$ .
  - Find  $\lim_{x \rightarrow 0^+} f(x)$  and  $\lim_{x \rightarrow 0^+} f'(x)$ .
  - Sketch the graph of  $f(x)$  using the information obtained in parts (a), (b), and (c). Clearly show the concavity of the graph and its behavior near the origin.
- C2.** [GS] Let  $m$  and  $n$  be positive integers and let  $x$  and  $y$  be positive real numbers such that  $x + y = S$ , where  $S$  is constant. Find expressions for the values of  $x$  and  $y$  that give the maximum possible value of the product  $P = x^m y^n$ .
- C3.** [CG] Suppose  $f(x)$  is a continuous increasing function for all positive real numbers  $x$ . Let  $a > 0$ . Then further suppose that  $t$  is the tangent line of  $f(x)$  at the point  $(a, f(a))$ , and that  $n$  is the normal line of  $f(x)$  at the point  $(a, f(a))$ . Let  $t$  intersect the  $x$ -axis at  $(x_t, 0)$  and  $n$  intersect the  $x$ -axis at  $(x_n, 0)$ . Let  $R$  represent the area of the triangle formed by connecting the points  $(a, f(a))$ ,  $(x_t, 0)$ , and  $(x_n, 0)$ .
- Find an expression, in terms of  $f(a)$  and  $f'(a)$ , for  $R$ .
  - Let  $f(x) = x^3 + 1$  and  $a = 1$ . Find intersection coordinates  $x_t$  and  $x_n$ , and then find area of the triangle formed by the points  $(1, f(1))$ ,  $(x_t, 0)$ , and  $(x_n, 0)$ .
  - What would be different about your expression in part (a) if  $f(x)$  was a continuous decreasing function?
- C4.** [AP72] Let  $f(x) = 4x^3 - 3x - 1$ .
- Find the  $x$ -intercepts of the graph of  $f$ .
  - Write an equation for the tangent line to the graph of  $f$  at  $x = 2$ .
  - Write an equation of the graph that is a reflection across the  $y$ -axis of the graph of  $f$ .

- C5.** [AP76] Let  $f(x) = x^3 - 3x^2 - 4x + 12$ . Define  $h(x)$  as

$$h(x) = \begin{cases} \frac{f(x)}{x-3} & x \neq 3 \\ p & x = 3 \end{cases}$$

where  $p$  is a constant.

- Find all zeros of  $f$ .
  - Find the value of  $p$  so that the function  $h$  is continuous at  $x = 3$ . Justify your answer.
  - Using the value of  $p$  found in part (b), determine whether  $h$  is an even function. Justify your answer.
- C6.** [AP76] Consider the hyperbola  $3x^2 - y^2 = 23$ .
- A point moves on the hyperbola so that its  $y$ -coordinate is increasing at a constant rate of 4 units per second. How fast is the  $x$ -coordinate changing when  $x = 4$ ?
  - For what values of  $k$  will the line  $2x + 9y + k = 0$  be normal to the hyperbola?
- C7.** [Arco] The vertices of a triangle are  $(0, 0)$ ,  $(x, \cos x)$ , and  $(\sin 3x, 0)$ , where  $0 < x < \pi/2$ .
- If  $A(x)$  represents the area of the triangle, write a formula for  $A(x)$ .
  - Find the value of  $x$  for which  $A(x)$  is maximum. Justify your answer.
  - What is the maximum area of the triangle?
- C8.** [Arco] Let  $p$  and  $q$  be constants so that  $q \neq 0$ . Define a function  $g(x)$  as

$$g(x) = \begin{cases} \frac{4x^2 - 1}{2x - 1} & x > 0 \\ qx + p & x = 0. \end{cases}$$

- Find all values of  $x$  for which  $g(x) = 0$ .
  - Find all points of discontinuity of the graph of  $g(x)$ .
  - Find the values for  $p$  and  $q$  so that  $g(x)$  is continuous at  $x = 0$ . Justify your answer.
  - Given that  $w > 0$ , find  $w$  such that  $\int_2^w g(x) dx = 6$ .
- C9.** [SH] A circular paper disc has a diameter of 8 inches. A sector with a central angle of  $x$  radians is cut out, and the sides of the remaining sector of angle  $2\pi - x$  are taped together to form a conical drinking cup. Find the angle  $x$  which results in a cone of maximum volume.



## Challenge Problem Set D

THESE PROBLEMS MAY BE COMPLETED AND TURNED IN AT ANY TIME BEFORE 3:05 PM, FEBRUARY 9

**D1.** [GS] Find the derivative of each of the following functions.

(a)  $y = (\ln x)^x$

(b)  $y = x^{\ln x}$

(c)  $y = (\ln x)^{\ln x}$

(d)  $y = x^{\sqrt{x}}$

**D2.** [SS] Let the function  $f$  be defined for all  $x$  and assume that  $f$  has a continuous derivative.

If  $f(0) = 0$  and  $0 < f'(x) \leq 1$ , then prove that  $\left[ \int_0^t f(x) dx \right]^2 \geq \int_0^t [f(x)]^3 dx$ .

**D3.** [RP] Let  $d > 0$ . Let  $A$  be the area bounded by  $y = x^n$  (for  $n > 1$ ) the  $x$ -axis, and  $x = 1 + d$ . Further, let  $V$  be the volume of the solid generated by rotating  $A$  about the  $x$ -axis.

(a) Find  $A$  and  $V$  as functions of  $n$  and  $d$ .

(b) Show that both  $\lim_{n \rightarrow 8} A = 8$  and  $\lim_{n \rightarrow 8} V = 8$ .

(c) Show that  $\lim_{n \rightarrow 8} \frac{A}{V} = \frac{2}{\pi}$  if  $d = 0$ , and that  $\lim_{n \rightarrow 8} \frac{A}{V} = 0$  if  $d > 0$ .

**D4.** [SH] If  $x \sin(\pi x) = \int_0^{x^2} f(t) dt$ , where  $f$  is a continuous function, then find  $f(4)$ .

**D5.** [SH] A region  $R$  is bounded by  $y = 1 + \sin(\frac{\pi x}{2})$ ,  $y = x/2$ , and the  $y$ -axis.

(a) Find the exact area of  $R$ .

(b) Find the exact volume of the solid obtained by revolving  $R$  about the  $x$ -axis.

(c) Find an expression involving an integral for the total perimeter of  $R$ .

**D6.** [SH] Let  $f$  and  $g$  be functions that are differentiable for all real numbers  $x$  and that have the following properties.

I.  $f'(x) = f(x) - g(x)$ .

II.  $g'(x) = g(x) - f(x)$ .

III.  $f(0) = 5$ .

IV.  $g(0) = 1$ .

(a) Prove that  $f(x) + g(x) = 6$  for all  $x$ .

(b) Find  $f(x)$  and  $g(x)$ . Show your work.

**D7.** [SH] At time  $t > 0$ , the volume of a sphere is increasing at a rate proportional to the reciprocal of its radius. At  $t = 0$ , the radius of the sphere is 1 and at  $t = 15$ , the radius is 2.

- (a) Find the radius of the sphere as a function of  $t$ .  
(b) At what time  $t$  will the volume of the sphere be 27 times its volume at  $t = 0$ ?

**D8.** [CG] Let  $f(x) = \int_0^{x^2} \frac{\sqrt{1+t^2}}{2\sqrt{t}} dt$ . Without using a calculator, find any extreme points and inflection points of  $f(x)$ , then sketch its graph.

**D9.** [SH] Consider the area in the first quadrant bounded by  $y = x^n$  ( $n > 1$ ), the  $x$ -axis, and the tangent line to  $y = x^n$  at the point  $(1, 1)$ . Find the value of  $n$  so that the enclosed area is a maximum.

**D10.** [DS] The cooling system in my old truck holds about 10 liters of coolant. Last summer, I flushed the system by running tap water into a tap-in on the heater hose while the engine was running and simultaneously draining the thoroughly mixed fluid from the bottom of the radiator. Water flowed in at the same rate that the mixture flowed out – at about 2 liters per minute. The system was initially 50% antifreeze. If we let  $W$  be the amount of water in the system after  $t$  minutes, then it follows that

$$\frac{dW}{dt} = 2 - 2\left(\frac{W}{10}\right).$$

- (a) Explain why the equation above is the correct one.  
(b) Find  $W$  as a function of time.  
(c) How long should I have let water run into the system to ensure that 95% of the mixture was water?

## Challenge Problem Set E

THESE PROBLEMS MAY BE COMPLETED AND TURNED IN AT ANY TIME BEFORE 3:05 PM, MARCH 23

NO CALCULATOR IS ALLOWED FOR ANY OF THESE PROBLEMS.

**E1.** [RP] Find the following antiderivatives.

$$\text{a) } \int \sqrt{\frac{1}{(x-2)^2} - \frac{1}{(x-4)^2}} dx \quad \text{b) } \int \frac{\ln(1 + \sin x)}{\cos^2 x} dx \quad \text{c) } \int e^{x^2} \ln(1 + e^{x^2})^x dx$$

**E2.** [RP] Evaluate both  $\lim_{x \rightarrow \infty} (e^{-x})^{1/\ln x}$  and  $\lim_{x \rightarrow 0^+} (e^x - 1)^{-1/\ln x}$ .

**E3.** [SH] Suppose  $f(x)$  is defined as  $f(x) = \begin{cases} 2x + 1 & x \leq 2 \\ \frac{1}{2}x^2 + k & x > 2. \end{cases}$

- (a) For what value of  $k$  will  $f$  be continuous at  $x = 2$ ? Justify your answer.
- (b) Using the value of  $k$  found in part (a), determine whether  $f$  is differentiable at  $x = 2$ .
- (c) Let  $k = 4$ . Determine whether  $f$  is differentiable at  $x = 2$ . Justify your answer.

**E4.** [SH] Let  $S$  be the series  $\sum_{n=0}^{\infty} \frac{t^n}{(1+t)^n}$ .

- (a) Find the value to which  $S$  converges when  $t = 1$ .
- (b) Determine the values of  $t$  for which  $S$  converges. Justify your answer.
- (c) Find all values of  $t$  that make the sum of the series  $S$  greater than 10.

**E5.** [SH]

- (a) A solid constructed so that it has a circular base of radius  $r$  centimeters and every plane section perpendicular to a certain diameter of the base is a square. Find the volume of the solid.
- (b) If the solid described in part a expands so that the radius of the base increases at a constant rate of 0.5 centimeters per minute, how fast is the volume changing when the radius is 4 centimeters?

**E6.** [GS] Solve the following initial-value problems.

$$\text{(a) } \frac{dy}{dx} = \frac{x(1+y^2)^2}{y(1+x^2)^2}, \quad y(2) = 1.$$

$$\text{(b) } \frac{dy}{dx} = \sqrt{xy - 4x - y + 4}, \quad y(5) = 8.$$

**E7.** [GS] Show that:

- (a)  $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$  converges. *Hint:* Express  $(\ln n)^{\ln n}$  as a power of  $n$ .
- (b)  $\sum_{n=2}^{\infty} \frac{1}{(\ln \ln n)^{\ln n}}$  converges.
- (c)  $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln \ln n}}$  diverges. *Hint:*  $(\ln \ln n)^2 \leq \ln n$  for large  $n$  (why?).
- (d)  $\sum_{n=3}^{\infty} \frac{1}{n \ln n \ln \ln n}$  diverges.
- (e)  $\sum_{n=3}^{\infty} \frac{1}{n \ln n (\ln \ln n)^p}$  converges if  $p > 1$  and diverges if  $p = 1$ .

**E8.** [GS] Calculate the following limits using Maclaurin series.

(a)  $\lim_{x \rightarrow 0} \frac{x \cos x - \sin x}{x^2 \tan x}$       (b)  $\lim_{x \rightarrow 0} \frac{\sin x - \tan x}{\sin 2x}$       (c)  $\lim_{x \rightarrow 0} \frac{\sqrt{1+x^2} + \cos x - 2}{x^4}$

**E9.** [SH] Let  $f$  a differentiable function defined for all  $x > 0$  such that the following properties hold.

- I.  $f(1) = 0$ .  
 II.  $f'(x) = 1$ .  
 III. The derivative of  $f(2x)$  equals  $f'(x)$  for all  $x$ .

- (a) Find  $f'(2)$ .  
 (b) Suppose  $f'$  is differentiable. Prove that there is a number  $c$ ,  $2 < c < 4$ , such that  $f''(c) = -\frac{1}{8}$ .  
 (c) Prove that  $f(2x) = f(2) + f(x)$  for all  $x > 0$ .

**E10.** [JS] For a fish swimming at a speed  $v$  relative to the water, the energy expenditure per unit time is proportional to  $v^3$ . It is believed that migrating fish try to minimize the total energy required to swim a fixed distance. If the fish are swimming against a current  $u < v$ , then the time required to swim a distance  $L$  is  $\frac{L}{v-u}$ , and the total energy  $E$  required to swim the distance is given by  $\frac{av^3L}{v-u}$ , where  $a$  is the proportionality constant.

- (a) Determine the value of  $v$  that minimizes  $E$ .  
 (b) Sketch the graph of  $E$ .

**Challenge Problem Set F**

THESE PROBLEMS MAY BE COMPLETED AND TURNED IN AT ANY TIME BEFORE 3:05 PM, MAY 4

NO CALCULATOR IS ALLOWED FOR ANY OF THESE PROBLEMS.

*All problems are taken from the 1957 AP Calculus Exam.*

- F1.** Prove that the ellipse  $2x^2 + y^2 = 6$  and the parabola  $y^2 = 4x$  intersect at right angles and sketch the curves, showing their points of intersection.
- F2.**  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$  are two points on the curve  $y = ax^2 + bx + c$ , with  $a \neq 0$ . A line is drawn parallel to the chord  $P_1P_2$  and tangent to the curve at the point  $P(x_0, y_0)$ . Prove that  $x_0 = \frac{x_1 + x_2}{2}$ .
- F3.** Find the equations of the straight lines through the point  $(-1, 0)$  which are tangent to the curve whose equation is  $y = x^2 - 3x$ . Sketch the curve and the tangents. Then prove that there is no tangent to curve through the point  $(2, 0)$ .
- F4.** A right circular cylinder of radius 12 inches is cut by a plane which passes through a diameter of the base and makes an angle of 60 degrees with the base. Set up an integral which will determine the volume of the wedge cut off. Then evaluate the integral.
- F5.** In the isosceles triangle  $ABC$ , the length of each of the sides  $AB$  and  $AC$  is 10 units and the length of  $BC$  is 12 units. Find the location of the point  $D$  on the altitude from  $A$  to  $BC$  for which the sum of the distances from  $D$  to the three vertices of the triangle is a minimum.
- F6.** A cylindrical tank whose radius is 10 feet and whose height is 25 feet is full of water. A leak occurs in the bottom of the tank and water escapes at a rate proportional to the square root of the depth of the water remaining in the tank. At the end of the first hour the depth of the remaining water is 6 feet. Obtain a formula for the depth of the water remaining after  $t$  hours.
- F7.** The area to the right of the line  $x = 1$  and inside the ellipse  $x^2 + 2y^2 = 9$  is revolved around the  $y$ -axis, thus generating a solid. Find its volume.
- F8.** Given a function  $f(x)$  defined for all real  $x$ , and such that  $f(x+h) - f(x) < 6h^2$  for all real  $h$  and  $x$ . Show that  $f(x)$  is a constant.
- F9.** Let  $f(x) = \int_1^x \frac{dt}{t + \sqrt{t^2 + 1}}$ , for  $x > 1$ . Is the following statement true or false?

$$\frac{1}{2}(\ln x) \leq f(x) \leq \ln x$$

Justify your answer.



---

## Challenge Problem Sources

---

- [AP72] Advanced Placement Calculus AB Exam, 1972
- [AP76] Advanced Placement Calculus AB Exam, 1976
- [Arco] *Arco's AP Exams in Mathematics: Calculus AB and Calculus BC*, Arco, 1990.
- [CG] Garner, Chuck, an original problem!
- [SH] Halfacre, Sandra, White Station High School AP Calculus website, White Station, Tennessee; <http://www.mecca.org/~halfacre/MATH/calc.htm>
- [AO] Ostebee, Arnold and Paul Zorn, *Calculus from Graphical, Numerical, and Symbolic Points of View*, 1st ed., Harcourt, 1997.
- [RP] Palmaccio, Richard, *Problems in Calculus*, SCT Publishing, 1985.
- [GS] Simmons, George, *Calculus with Analytic Geometry*, 2nd ed., McGraw-Hill, 1996.
- [DS] Smith, David A. and Lawrence Moore, *Calculus: Modeling and Application*, D.C. Heath, 1996.
- [SS] Stein, Sherman and Anthony Barcellos, *Calculus and Analytic Geometry*, 5th ed., McGraw-Hill, 1992.
- [JS] Stewart, James, *Calculus*, 4th edition, Brooks-Cole, 1999.
- [TICAP] *Technology Intensive Calculus for Advanced Placement*, workshop, 1995.





# APPENDIX A

---

## FORMULAS

---

## Formulas from Geometry

### Area Formulas

#### Square

$$A = s^2 \text{ where } s \text{ is the side length}$$

$$A = \frac{1}{2}d^2 \text{ where } d \text{ is the length of the diagonal}$$

#### Triangle

$$A = \frac{1}{2}bh \text{ where } b \text{ is the base and } h \text{ is the altitude}$$

$$A = \sqrt{s(s-a)(s-b)(s-c)} \text{ where } s \text{ is the semiperimeter and } a, b, \text{ and } c \text{ are the sides}$$

$$A = sr \text{ where } s \text{ is the semiperimeter and } r \text{ is the radius of the inscribed circle}$$

$$A = \frac{1}{2}ab \sin \theta \text{ where } a \text{ and } b \text{ are two sides and } \theta \text{ is the measure of the angle between } a \text{ and } b$$

#### Equilateral Triangle

$$A = \frac{1}{4}s^2\sqrt{3} \text{ where } s \text{ is the side length}$$

$$A = \frac{1}{3}h^2\sqrt{3} \text{ where } h \text{ is the altitude}$$

#### Parallelogram

$$A = bh \text{ where } b \text{ is the base and } h \text{ is the altitude}$$

#### Rhombus

$$A = bh \text{ where } b \text{ is the base and } h \text{ is the altitude}$$

$$A = \frac{1}{2}d_1d_2 \text{ where } d_1 \text{ and } d_2 \text{ are the two diagonals}$$

#### Kite

$$A = \frac{1}{2}d_1d_2 \text{ where } d_1 \text{ and } d_2 \text{ are the two diagonals}$$

#### Trapezoid

$$A = \frac{1}{2}(b_1 + b_2)h \text{ where } b_1 \text{ and } b_2 \text{ are the parallel bases and } h \text{ is the distance between them}$$

#### Cyclic Quadrilateral

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)} \text{ where } s \text{ is the semiperimeter and } a, b, c, d \text{ are the sides}$$

#### Regular Polygon

$$A = \frac{1}{2}ans \text{ where } a \text{ is the apothem, } n \text{ is the number of sides, and } s \text{ is the side length}$$

$$A = \frac{1}{2}ap \text{ where } a \text{ is the apothem and } p \text{ is the perimeter}$$

#### Ellipse

$$A = ab\pi \text{ where } a \text{ is half the major axis and } b \text{ is half the minor axis}$$

#### Circle

$$A = \pi r^2 \text{ where } r \text{ is the radius}$$

$$A = \frac{1}{2}Cr \text{ where } C \text{ is the circumference and } r \text{ is the radius}$$

$$A = \frac{1}{4}\pi d^2 \text{ where } d \text{ is the diameter}$$

#### Sector of a Circle

$$A = \frac{1}{360^\circ}\pi ar^2 \text{ where } a \text{ is the angle (in degrees) that intercepts the arc and } r \text{ is the radius}$$

$$A = \frac{1}{2}ar^2 \text{ where } a \text{ is the angle (in radians) that intercepts the arc and } r \text{ is the radius}$$

**Surface Area Formulas***Prism and Cylinder*

$S = 2B + ph$  where  $B$  is the area of the base,  $p$  is the perimeter of the base, and  $h$  is the height

*Pyramid and Cone*

$S = B + \frac{1}{2}ps$  where  $B$  is the area of the base,  $p$  is the perimeter of the base, and  $s$  is the slant height of a lateral face

*Sphere*

$S = 4\pi r^2$  where  $r$  is the radius

**Volume Formulas***Prism and Cylinder*

$V = Bh$  where  $B$  is the area of the base and  $h$  is the height

*Pyramid and Cone*

$V = \frac{1}{3}Bh$  where  $B$  is the area of the base and  $h$  is the height

*Sphere*

$V = \frac{4}{3}\pi r^3$  where  $r$  is the radius

**Greek Alphabet**

Upper case	Lower case		Upper case	Lower case	
A	$\alpha$	alpha	N	$\nu$	nu
B	$\beta$	beta	$\Xi$	$\xi$	xi
$\Gamma$	$\gamma$	gamma	O	o	omicron
$\Delta$	$\delta$	delta	$\Pi$	$\pi$	pi
E	$\epsilon$	epsilon	R	$\rho$	rho
Z	$\zeta$	zeta	$\Sigma$	$\sigma$	sigma
H	$\eta$	eta	T	$\tau$	tau
$\Theta$	$\theta$	theta	$\Upsilon$	$\upsilon$	upsilon
I	$\iota$	iota	$\Phi$	$\phi$	phi
K	$\kappa$	kappa	X	$\chi$	chi
$\Lambda$	$\lambda$	lambda	$\Psi$	$\psi$	psi
M	$\mu$	mu	$\Omega$	$\omega$	omega

## Trigonometric Values

$$\sin 0 = 0$$

$$\sin \frac{\pi}{6} = \frac{1}{2}$$

$$\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\cos 0 = 1$$

$$\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\tan 0 = 0$$

$$\tan \frac{\pi}{6} = \frac{\sqrt{3}}{3}$$

$$\tan \frac{\pi}{4} = 1$$

$$\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

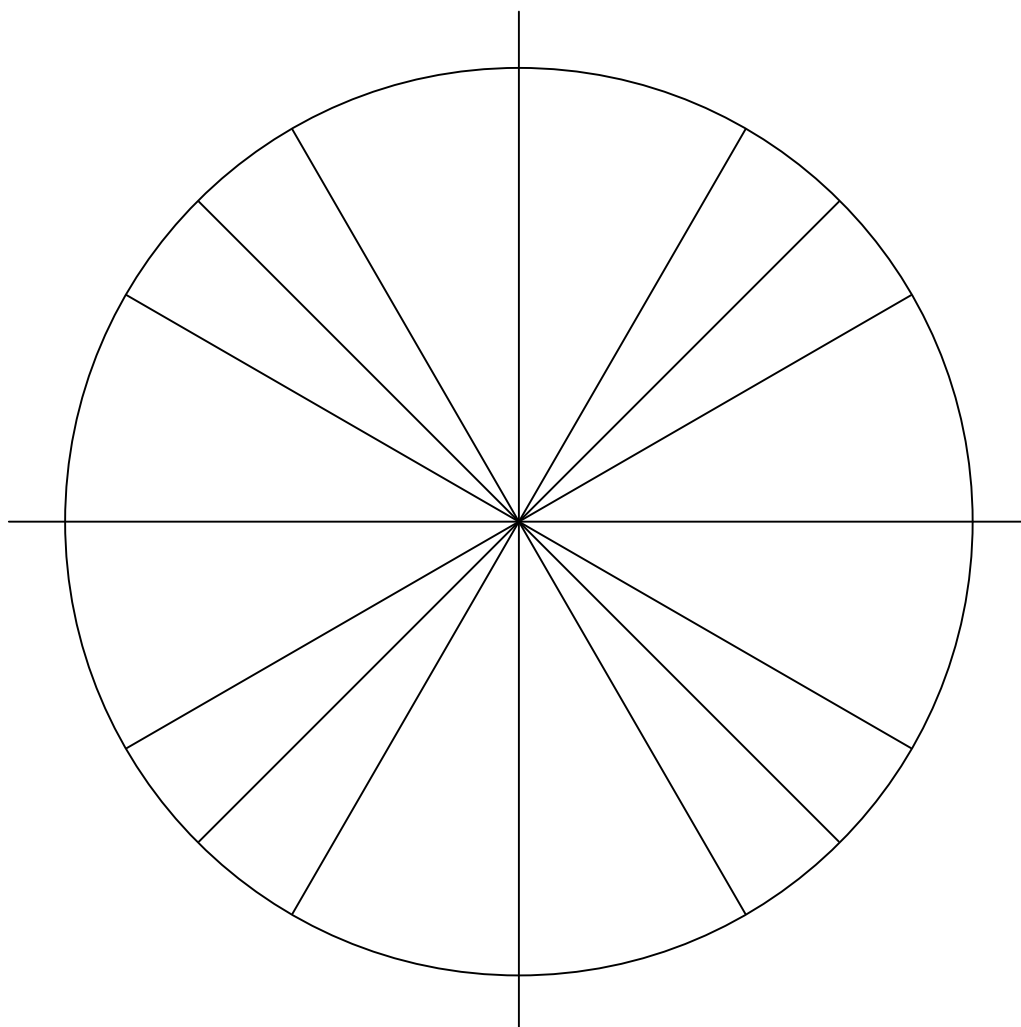
$$\sin \frac{\pi}{2} = 1$$

$$\cos \frac{\pi}{3} = \frac{1}{2}$$

$$\cos \frac{\pi}{2} = 0$$

$$\tan \frac{\pi}{3} = \sqrt{3}$$

$$\tan \frac{\pi}{2} \text{ is undefined}$$



## Useful Trigonometric Identities

### Triangle Ratios

$$\sin x = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos x = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\csc x = \frac{1}{\sin x} = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\sec x = \frac{1}{\cos x} = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\cot x = \frac{\cos x}{\sin x} = \frac{\text{adjacent}}{\text{opposite}}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{\text{opposite}}{\text{adjacent}}$$

### Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

$$\cot^2 x + 1 = \csc^2 x$$

### Double Angle Identities

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2 \sin^2 x = 2 \cos^2 x - 1$$

### Power Identities

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

### Sum and Difference Identities

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

### Law of Cosines

$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$

### Law of Sines

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}$$



## APPENDIX B

---

# **SUCCESS IN MATHEMATICS**

---

## Calculus BC Syllabus

**Instructor:** Dr. Chuck Garner

**Contact Information:**

e-mail [cgarner@rockdale.k12.ga.us](mailto:cgarner@rockdale.k12.ga.us)

AIM name: DocCoachG

**Texts:**

1. Garner, Chuck. *The AP Calculus Problem Book*, 4th edition, revised and corrected, RMSST, 2008. \$15.00

**Additional Resources:**

1. Thomas, George B., Ross L. Finney, Maurice D. Weir, and Frank R. Giordano. *Thomas' Calculus, Early Transcendentals Part One: Single Variable*, Updated 10th edition, Addison-Wesley, 2003. ISBN 0321123565. \$102.96
2. Lederman, David. *Multiple-Choice and Free-Response Questions in Preparation for the AP Calculus BC Examination*, 6th edition, D & S Marketing Systems, 1999. \$21.95
3. Thomas, George B. and Ross L. Finney. *Student's Study Guide, Part One*, Addison-Wesley, 2001. ISBN 0201710080. \$28.80
4. Weir, Maurice D. and John Scharf. *Student's Solutions Guide, Part One*, Addison-Wesley, 2001. ISBN 0201662116. \$27.60
5. The *Visual Calculus* Web Page: <http://archives.math.utk.edu/visual.calculus>
6. The Magnet Math Website: <http://drgarnerjr.home.comcast.net>

**Course Description:** This course is designed to provide a college-level experience in mathematics. Students will be able to work with functions in numerical, graphical, and algebraic ways and will also understand the relationships between the different representations. Students will understand the concepts of limits, derivatives, and integrals. Students will be able to apply derivatives and integrals to real-world phenomena. Students will understand and apply differential equations, polar functions, vector functions, sequences, and series. Broad goals for the students include understanding the role of calculus concepts in science and technology; being more than adequately prepared for the AP Calculus exam; and, developing an interest and appreciation for mathematics itself, outside scientific and technological applications.

**Materials:**

You will need the following materials on a daily basis.

1. a notebook for notes
2. a binder with loose-leaf paper for homework
3. pencils (any assignment to be graded must be completed in pencil)
4. a graphing calculator (TI-83 or TI-84 is recommended)
5. the *Problem Book*

**Requirements:**

1. Bring all materials. *Note: I do not provide extra materials if you fail to bring your materials.*
2. Complete all assignments.
3. Use class time constructively.
4. Actively participate in class discussions.
5. A desire to learn and a determination to succeed.



**Letter Grades:** 90 to 100: A; 80 to 89: B; 75 to 79: C; 70 to 74: D; and below 70: F.

*Grades are confidential and will only be discussed with the concerned student and the student's parents.*

**Evaluation:**

<i>First Semester.</i>		<i>Second Semester.</i>	
3 Tests, each 100 pts	300	4 Tests, each 100 pts (drop 1)	300
10 Quizzes, each 20 pts (drop 1)	180	7 Quizzes, each 20 pts (drop 1)	120
6 Labs, each 40 pts (drop 1)	200	4 Labs, each 40 pts (drop 1)	120
5 TI-Labs, each 10 pts (drop 1)	40	3 TI-Labs, each 10 pts	30
37 Homeworks, each 10 pts (drop 2)	350	37 Homeworks, each 10 pts (drop 2)	350
Portfolio	130	Project	180
<b>TOTALS:</b>	<b>1200</b>		<b>1000</b>

The following formula is used to determine your grade in this class each Semester:

$$\frac{\text{Total points earned}}{\text{Total points possible}}$$

As always, the Final Grade for each semester is computed by

$$\text{Final grade} = 0.8(\text{Semester grade}) + 0.2(\text{Exam Grade})$$

Students are required to keep track of their own grades. You may compare your grade calculations with me after school; I will not discuss grades during the school day.

**Practice AP Exam:** EITHER Sunday, April 19 OR Sunday, April 26 at 1:00 PM.  
**AP EXAM:** Wednesday, May 6 at 8:00 AM.

**Make-Up Work:**

**Tests** No make-up tests will be given unless arrangements are made prior to test day. If a test is missed first semester, the final exam will be used to replace the missed test. The final exam will not be used to replace any test grades second semester. There are no exam exemptions. If your test average over both semesters is an A, you are exempt from the second semester Project.

**Quizzes** No make-up quizzes will be given.

**Homework** No make-up homework assignments will be given.

**Extra Credit** No extra credit of any kind will be given.

**Absences** It is entirely the student's responsibility to obtain notes, handouts, and assignments when the student is absent.

*Cheating of any kind on any assignment is considered the theft of someone else's diligence will result in zero points for that assignment for all persons involved and possibly a grade of "F" for the course.*

**This syllabus provides a general plan for the course; deviations may be necessary.**

### BC Schedule First Semester

Day	Assignment	Day	Assignment
1	1.1	21	2.18, 2.19
2	1.2, 1.3, TI-Lab 1 due		TEST 2
3	1.4, 1.5	22	3.1, 3.2
4	Quiz, 1.6, 1.7	23	3.3, 3.4, Calculus Lab 4 due
5	1.8–1.10	24	Quiz, 3.5, 3.6
6	1.11, 1.12, Calculus Lab 1 due	25	3.7
7	1.13, 1.14	26	3.8, TI-Lab 3 due
8	Quiz, 1.15–1.17	27	3.9, 3.11
9	1.18, 1.19	28	Quiz, 3.13, 3.14
	TEST 1		TEST 3
10	2.1, 2.2	29	4.1, 4.2
11	Quiz, 2.3, Calculus Lab 2 due	30	4.3
12	2.4, 2.5	31	4.4, 4.5, Calculus Lab 5 and TI-Labs 4 and 5 due
13	2.6, 2.7	32	4.6
14	2.8, TI-Lab 2 due	33	Quiz, 4.7, 4.8
15	Quiz, 2.9, 2.10	34	4.9
16	2.11, 2.12	35	4.10, 4.11, Portfolio due
17	2.13, Calculus Lab 3 due	36	4.12, 4.13, Calculus Lab 6 and TI-Lab 6 due
18	Quiz, 2.14	37	Quiz, 4.14–4.16
19	2.15, 2.16	38	Quiz, 4.17–4.19
20	2.17		EXAM

**BC Schedule Second Semester**

Day	Assignment		
		58	7.9
39	5.1	59	7.10
40	5.2	60	7.11, 7.12
41	5.3	61	Quiz, 7.13
42	Quiz, 5.4, TI-Lab 7 due	62	7.14
43	5.5, 5.6	63	7.15, 7.16
44	5.7, 5.8		TEST 5
45	Quiz, 6.1, TI-Lab 8 due	64	Exam II Section IIA
46	6.2	65	Exam II Section IIB
47	6.3, 6.4	66	Exam III Section IIA
48	Quiz, 6.5, Calculus Lab 7 due	67	Exam III Section IIB
49	6.6	68	Exam IV Section IIA
50	6.7, 6.8	69	Exam IV Section IIB
51	Quiz, 5.9, 5.10, 6.9, 6.10	70	Quiz, Exam V Section IIA
	TEST 4	71	Exam V Section IIB
52	7.1, 7.2, Calculus Lab 8 and TI-Lab 9 due		Tues. May 5: Project due
			Wed. May 6: AP EXAM, 8:00 AM
53	7.3	72	8.1
54	7.4	73	8.2
55	7.5	74	8.3
56	Quiz, 7.7, Calculus Lab 9 due	75	8.4, Calculus Lab 10 due
57	7.8		EXAM, Group Investigation due

## How to Succeed in This Class

(Adapted from an essay by Dan Kennedy)

Here are some facts: I want you to succeed. You want yourself to succeed. We should obviously be able to work this out to our mutual satisfaction. What you are about to read is an effort to work this out.

One measure of success, perhaps unfortunately, is your grade. A grade is an inadequate way to measure achievement, but a fairly realistic way to measure performance. Indeed, your performance is always being evaluated by someone or other: a parent, a friend, an employer, or even a casual observer. A teacher simply takes this evaluation a step further and puts a grade on it.

I will measure and grade your performance based on four things: *DILIGENCE*, *KNOWLEDGE*, *COLLABORATION*, and *CLEVERNESS*. (All academic grades are probably based on the same criteria, whether the grader admits it or not.) Note that only *cleverness* has anything to do with the relative size of your brain, so a grade will not necessarily measure how smart you are.

Your grade will be based on six types of performance: classwork, homework, labs, tests, quizzes, and a portfolio/project. The specifics of grading the labs can be found elsewhere in this book.

### CLASSWORK

Everyone must have a notebook that will be devoted exclusively to classwork for this class. Don't worry about filling it up; we will. (Homework should be done on paper from another source.) You must always bring this notebook to class, along with a pencil and a graphing calculator. When class begins, your notebook, pencil, and calculator should be on the table, ready for action. Nothing else should be on your table while you are in this class. Sometimes you will copy what I write on the board; sometimes you will work problems alone; sometimes you will work in groups. Whatever we do, however, you will keep a record of it in the notebook.

Your notebook will not be collected, but it will probably be a source of some of your best *portfolio* items (see below). Meanwhile, I will be evaluating how well you and your classmates work together through personal observation over the course of the semester. How well do you work together with others in groups? (This is the *collaboration* part.) How involved are you in class discussions? How well can you ask a question when you don't understand, and how well can you explain it to others when you do understand? Do you come for extra help when you need it? Does your attitude help the class or hinder it? Based on these criteria I will arrive at a subjective opinion of the class's overall quality, which will determine how grades are curved for your class. (See *TESTS* below for more on the curve.)

### HOMEWORK

There will be a short homework assignment almost every night. You must do mathematics in order to understand it, so doing these assignments is essential. This is also the main opportunity I will have in my grading scheme to reward *diligence*, so students who feel that they might be lacking in the *cleverness* category should be especially attentive to homework.

Regardless of the assignment, there are basically two steps to doing any math problem:

Step 1: Find out how to do the problem.

Step 2: Do the problem.

The distinction between these two steps, while subtle, is probably the key to success in this course. There are no restrictions whatsoever on how you accomplish the first of these two steps! If you know how to do the problem, fine: Do it. If you do not know how to do the problem then you are at Step 1, and I encourage you to get help wherever you can find it. Read the book. Check your notebook. Talk to a classmate. See me for extra help. Ask your mother. Post a question on a forum or blog. Pay a tutor. Different people

have different learning styles, and what works for someone else might not work for you. On the other hand, it is your education, and you need to find something that works. Collaboration is *good for you*, as long as it is in Step 1.

Now, once you have learned how to do the problem – this is very important – you are at Step 2, which you must do on your own! That is the performance part, and remember: that is the part that I am grading. If you *COPY* somebody else’s homework, then you are (a) cheating for a grade, and (b) defeating the whole learning system by proceeding to Step 2 without ever having confronted Step 1. The extent of such cheating is even greater when one considers that it is precisely the diligence of another student that is being stolen, and diligence is what I am intending to reward. I want very much to encourage you to work together to understand the material, but you will never understand it unless you do the mathematics on your own when it comes time to do it. It is precisely this aspect of copying – the theft of someone else’s diligence – that will be a punishable offense.

In a similar vein, you are avoiding Step 1 if you simply copy an answer from the back of the book. Checking those answers is a good idea, but only insofar as it helps you to learn how to do the problem. (If your answer agrees, it is usually a good sign that you can do the problem.) *Be sure to show your work*, and be aware that the answer in the back of the book is not always right!

When you write up your homework, do it on standard  $8\frac{1}{2} \times 11$  paper, and keep in mind that it is a document by which you will be judged. Write legibly, spell words correctly, and make your steps clear. Don’t reinforce any bad habits that might lead you to sign your name to shoddy work some day when the stakes are higher. Fold the paper vertically, and with the crease on your left (like a book), put your name on the outside of the sheet. Details concerning writing up homework are found in the diagram at the end of this little essay.

Homework is due at the beginning of class. Late homework will be accepted up to nine days late, for a one-point penalty for each day missed. Please keep in mind that the standard for late homework is by *calendar day*, not by *class day*. Homework missed due to an excused absence can be handed in at no penalty, with a grace period of one day for each day of excused absence. Please note that *keeping track of homeworks missed due to absences will be the student’s responsibility*.

On most occasions, I will grade one student’s paper and then have that student grade the rest of the papers according to that key. Grades will be on a 10-point scale with increments of  $\frac{1}{2}$ . Students should take this grading responsibility seriously. Remember that this is the best chance for diligence to shine in my grading system, and laziness must not be rewarded. Partial credit may be awarded as the grader sees fit, just so long as it is fair to all.

### TESTS

Tests will be designed primarily to test *knowledge*, although *cleverness* is certainly useful, and *diligence* will have played a strong role in how much knowledge was accumulated prior to the test. My intent on tests is to find out what you know and to get you ready for the format and style of the AP Exam. Tests are graded according a standard rubric – a copy of which you will receive – and there is a penalty for guessing on multiple-choice questions.

Because major tests assess *knowledge*, the grades will be curved to reflect how much material the class has actually absorbed. A 90 on an easy test could actually reflect less knowledge than a 70 on a more demanding test, so curving to a subjective “class average” compensates for that. The better the class, the higher the class average; the higher the class average, the better the curve. Students who struggle in a class that is challenged can expect more benefit from my curve than students who are comfortable in a class that is coasting! In terms of *knowledge*, the students

in the better class are better off than the students in the lazy class.

There is consequently an important group component to each of my math classes. Each individual's performance is obviously significant, but the "class average" is based on how the class as a whole is doing. Is the class being dragged down by its lowest components, or being elevated by its highest components? Does the class work well together mathematically, or do some students resist work to the point of impeding the efforts of others? Does the class welcome challenges, or seek the path of least resistance? Am I the coach, or am I the enemy? Is it me against you or US against ignorance? The class as a whole must make those decisions somehow.

### PORTFOLIO

One valid criticism of traditional classroom assessments is that the students themselves have no significant input in determining what will be assessed. Essentially, teachers create the hoops for students to jump through, and students must hope that the hoops will match up well with what they have learned and how they can best communicate it. Portfolio assessment is an attempt to empower the student in his or her own assessment process. There will therefore be a portfolio grade first semester based on a portfolio of work by which you would like to be judged. Here is how it will work.

For your portfolio you must produce several pieces of your own work which you believe describe (positively) your development in this course. These could be based on tests, quizzes, homework, notebook entries, experiences outside of class, or your own feelings. They could take the form of reflective essays, poems, artistic works, or whatever. The important thing is that each portfolio entry should give me evidence of your learning that I otherwise would not have. A perfect quiz, while certainly good evidence of your learning, is not a good portfolio item – because it is evidence that already resides in my grade

book. A bad quiz could be a great portfolio item if it is accompanied by a reflective essay on why you did badly, with some proof that you subsequently mastered the material you did not know at the time. I would be glad to discuss any item with you before it goes into the portfolio. (This will provide an occasion for you to explain your selection to me, and for me to react to your choice in your presence.) Portfolio grades will not be factored in at midterm, but you ought to have at least one item in the portfolio by that time. You will need a minimum of *three* items by the end of the semester.

Since the intent of the portfolio is to communicate your learning to me in ways that go beyond what my gradebook already tells me, my assumption will be that the portfolio can only HELP your average, and it will be graded accordingly. However, if your portfolio is incomplete, missing, or carelessly shoddy, then your *homework* average will be reduced by up to 10 points for showing lack of diligence.

Due at the end of second semester is a project. The details of the project will be given just before Spring Break, and it will be due before the AP Exam.

### QUIZZES

Quizzes differ from tests in three important ways: (1) they are shorter – usually three to five problems; (2) they concentrate on less material than a test; and (3) they are intended to reward both knowledge and cleverness. This last point is significant philosophically, as it means that quiz questions are not necessarily fair assessments of what a student knows. Knowing the material does not guarantee success (although not knowing the material will probably guarantee a lack of success). Quiz questions will often require extra thought or insight. On quizzes, such comments as "You tried to trick us!" or "You never showed us anything like that!" or "How do you expect us to answer this if nobody gets it right?" are all irrelevant. What matters is that the question has an answer. If only

two students get a quiz question right, don't blame the question; instead, try to be one of those two students next time!

There will be occasional "partner quizzes" during the year to assess your performance in a collaborative setting. Partners will be randomly assigned, and both students in the pair will receive the same score. (Tests will never be collaborative.)

### THE CURVE

The class average begins at an 85. This is based on class averages of AP Calculus classes for the past few years.

You can raise the class average by exceeding my expectations and can lower the class average by disappointing me, but it is that class average that will determine the scaling of all tests and final exams. The better you are, the more you can expect me to challenge you and the better will be your chances of showing me how high your class average should be.

If a test is especially difficult for a class, then they are protected by the fact that the class average moves sluggishly: say from 85 down to 83. I can understand how an 85 class might become an 83 class in the few weeks between tests, but how could they suddenly plunge to 78, unless at least one of those tests was a faulty indicator of how good that class was? So, let us say that I give a challenging test to a class whose average stands at 91. They handle the stuff I expect them to handle, and several of them surprise me on the hard ones. They make the usual careless mistakes, but everyone is doing the right kind of mathematics. Grading on an AP scale, I find that the test average is 75. I look back on the homework effort for the past few weeks, the class participation, and so on, and I decide to raise the class average to 92. This gives me an ordered pair (75, 92) for scaling raw grades to real grades. Now suppose that my top student has managed a raw score of 93, some fantastic work, which I decide to scale to 99. That gives me a second ordered pair (93, 99). Those two points determine a linear equation

that enables me to scale anyone's grade in a fair and objective manner. Mathematically, the effect of this scaling is to adjust the mean (a primary goal) and to reduce the standard deviation (a secondary effect that helps me accomplish the primary goal of teaching mathematics to my entire class).

For example, let us suppose that this test really catches one student dismally unprepared, for any number of academic or other reasons. Say the student gets a raw score of 20. My scale brings that up to a 71, where it is still an outlier in terms of a much smaller standard deviation, but where the student can still believe that a comeback is possible. Notice that the class average is very significant here; if we change that class average to 82 rather than 92 and leave everything else the same, the raw score of 20 scales to a real score of only 30.

Remember: you and your classmates determine the class average.

### EXPECTATIONS

Enough about assessment. Now a few brief words about behavior and general expectations.

I can teach you this material. You, however, have to give me the chance to teach it to you. For that reason, your involvement in the class must be total and undivided, and I want you to pin me down with questions when you are confused. If you are not paying attention, you are hurting yourself. If you are distracting others, you are hurting them. If you are distracting me, then you are hurting everybody, and all these distractions affect the class average – which is the key to the curve. Your attitude can actually raise or lower everyone's grade by affecting the curve, and I want you all to FEEL that responsibility for each other's welfare. My grading process, my teaching style, and my entire educational philosophy are based on the premise that learning mathematics is a *group effort*. Colleges and universities owe their very existence to that fundamental premise. Conse-

quently, I consider all incidents of bad classroom behavior to be acts of selfishness more than anything else. Think carefully about the effect you have on the learning of others.

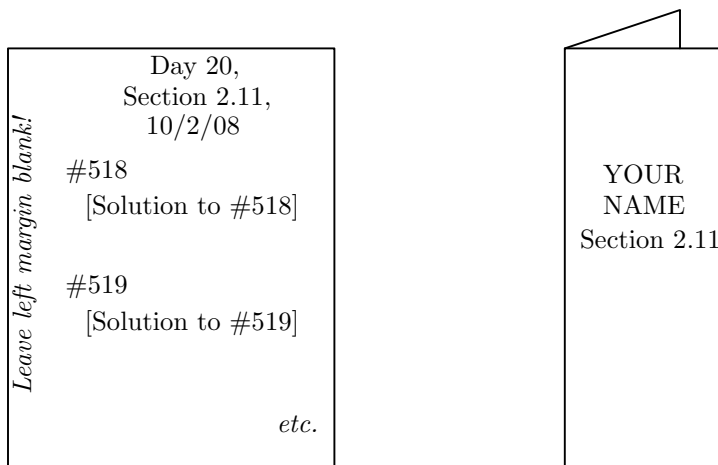
If you ever find yourself falling behind, get extra help! It does not even have to be help from me. *Find a classmate and work together!* If neither one of you can understand something, then you can *both* come to me and we'll help twice as many people.

Don't be absent. It's much easier to be here than to catch up after you have not been here. If you do get sick, leave space in your notebook for each day missed and fill in the gaps when you return.

On the next four pages you will find very useful suggestions for studying for a math class such as AP Calculus. It is worth your time to take heed of at least a few of those suggestions.

— Suggestions for submitting homework —

- ✓ Your handwriting should be legible.
- ✓ Homework with multiple pages should be stapled in the upper left-hand corner.
- ✓ In the top center of the first page, you should write the Day number, the assignment (section number and/or problem numbers) and the date it was assigned.
- ✓ Problems should be clearly labeled and numbered on the left side of the page. There should also be a visible separation between problems.
- ✓ You should leave the entire left margin blank so that graders may use this space for scoring and comments.
- ✓ Only write on the front side of the paper, never on the back.
- ✓ To ensure that each problem is graded, problems should be written in the order they are assigned.
- ✓ It is good practice to first work out the solutions to homework problems on scratch paper, and to then neatly write up your solutions. This will help you to turn in a clean finished product.
- ✓ To submit your homework, fold your papers together lengthwise like a book (the first page is on the inside and the last page is the outside) and write your name and the assigned section number clearly on the outside like the title of a book.





## Math Study Skills

The material is adapted from the Mathematics Department webpage of Saint Louis University, <http://euler.slu.edu/Dept/SuccessinMath.html>.

### Active Study vs. Passive Study

Be *actively* involved in managing the learning process, the mathematics, and your study time:

- Take responsibility for studying, recognizing what you do and don't know, and knowing how to get the Instructor to help you with what you don't know.
- Attend class every day and take complete notes. Instructors formulate test questions based on material and examples covered in class as well as on those in the book.
- Be an active participant in the classroom. Read ahead in the textbook; try to work some of the problems before they are covered in class. Anticipate what the Instructor's next step will be.
- Ask questions in class! There are usually other students wanting to know the answers to the same questions you have.
- Go to office hours and ask questions. The Instructor will be pleased to see that you are interested, and you will be actively helping yourself.
- Good study habits throughout the semester make it easier to study for tests.

### Studying Math is Different from Studying Other Subjects

- Math is learned by doing problems. It is vital that you *DO THE HOMEWORK*. The problems help you learn the formulas and techniques you do need to know, as well as improve your problem-solving prowess.

- A word of warning: Each class builds on the previous ones, all semester long. You must keep up with the Instructor: attend class, read the text and do homework every day. Falling a day behind puts you at a disadvantage. Falling a week behind puts you in deep trouble.
- A word of encouragement: Each class builds on the previous ones, all semester long. You're always reviewing previous material as you do new material. Many of the ideas hang together. *Identifying and learning the key concepts means you don't have to memorize as much.*

### College Math is Different from High School Math

A College math class covers material at about twice the pace that a High School course does. You are expected to absorb new material much more quickly. Tests are probably spaced farther apart and so cover more material than before. The Instructor may not even check your homework.

- Take responsibility for keeping up with the homework. Make sure *you* find out how to do it.
- You probably need to spend more time studying per week – you do more of the learning outside of class than in High School.
- Tests may seem harder just because they cover more material.

### Study Time

You may know a rule of thumb about math (and other) classes: at least 2 hours of study time per class hour. But this may not be enough!

- Take as much time as you need to do all the homework and to get complete understanding of the material.

- Form a study group. Meet once or twice a week (also use the phone, email, and instant messaging). Go over problems you've had trouble with. Either someone else in the group will help you, or you will discover you're all stuck on the same problems. Then it's time to get help from your Instructor.
- The more challenging the material, the more time you should spend on it.

### Studying for a Math Test

#### *Everyday Study is a Big Part of Test Preparation*

Good study habits throughout the semester make it easier to study for tests.

- *Do the homework when it is assigned.* You cannot hope to cram 3 or 4 weeks worth of learning into a couple of days of study.
- On tests you have to solve problems; homework problems are the only way to get practice. As you do homework, make lists of formulas and techniques to use later when you study for tests.
- Ask your Instructor questions as they arise; don't wait until the day or two before a test. The questions you ask right before a test should be to clear up minor details.

#### *Studying for a Test*

- Start by going over each section, reviewing your notes and checking that you can still do the homework problems (*actually work the problems again*). Use the worked examples in the text and notes – cover up the solutions and work the problems yourself. Check your work against the solutions given.

- *You're not ready yet!* In the book each problem appears at the end of the section in which you learned how to do that problem; on a test the problems from different sections are all together.
  - Step back and ask yourself what kind of problems you have learned how to solve, what techniques of solution you have learned, and how to tell which techniques go with which problems.
  - Try to explain out loud, in your own words, how each solution strategy is used. If you get confused during a test, you can mentally return to your verbal “capsule instructions.” Check your verbal explanations with a friend during a study session (it's more fun than talking to yourself!).
  - Put yourself in a test-like situation: work problems from review sections at the end of chapters, and work old tests if you can find some. It's important to keep working problems the whole time you're studying.
- Also:
  - Start studying early. Several days to a week before the test (longer for the final), begin to allot time in your schedule to reviewing for the test.
  - Get lots of sleep the night before the test. Math tests are easier when you are mentally sharp.

### Taking a Math Test

#### *Test-Taking Strategy Matters*

Just as it is important to think about how you spend your study time (in addition to actually doing the studying), it is important to think about what strategies you will use when you take a test (in addition to actually doing

the problems on the test). Good test-taking strategy can make a big difference to your grade!

### *Taking a Test*

- First look over the entire test. You'll get a sense of its length. Try to identify those problems you definitely know how to do right away, and those you expect to have to think about.
- Do the problems in the order that suits you! Start with the problems that you know for sure you can do. This builds confidence and means you don't miss any sure points just because you run out of time. Then try the problems you think you can figure out; then finally try the ones you are least sure about.
- Time is of the essence – work as quickly and continuously as you can while still writing legibly and showing all your work. If you get stuck on a problem, move on to another one – you can come back later.
- Show all your work: make it as easy as possible for the Instructor to see how much you do know. Try to write a well-reasoned solution. If your answer is incorrect, the Instructor will assign partial credit based on the work you show.
- Never waste time erasing! Just draw a line through the work you want ignored and move on. Not only does erasing waste precious time, but you may discover later that you erased something useful (and/or maybe worth partial credit if you cannot complete the problem). You are (usually) not required to fit your answer in the space provided - you can put your answer on another sheet to avoid needing to erase.
- In a multiple-step problem outline the steps before actually working the problem.

- Don't give up on a several-part problem just because you can't do the first part. Attempt the other part(s) – the actual solution may not depend on the first part!
- Make sure you read the questions carefully, and do all parts of each problem.
- Verify your answers – does each answer make sense given the context of the problem?
- If you finish early, check every problem (that means rework everything from scratch).

### **Getting Assistance**

#### *When*

Get help as soon as you need it. Don't wait until a test is near. The new material builds on the previous sections, so anything you don't understand now will make future material difficult to understand.

#### *Use the Resources You Have Available*

- Ask questions in class. You get help and stay actively involved in the class.
- Visit the Instructor's Office Hours. Instructors like to see students who want to help themselves.
- Ask friends, members of your study group, or anyone else who can help. The classmate who explains something to you learns just as much as you do, for he/she must think carefully about how to explain the particular concept or solution in a clear way. So don't be reluctant to ask a classmate.
- Find a private tutor if you can't get enough help from other sources.
- All students need help at some point, so be sure to get the help you need.

### Asking Questions

Don't be afraid to ask questions. Any question is better than no question at all (at least your Instructor/tutor will know you are confused). But a good question will allow your helper to quickly identify exactly what you don't understand.

- An unhelpful comment: "I don't understand this section." The best you can expect in reply to such a remark is a brief review of the section, and this will likely overlook the particular thing(s) which you don't understand.
- Good comment: "I don't understand why  $f(x+h)$  doesn't equal  $f(x)+f(h)$ ." This is a very specific remark that will get a very specific response and hopefully clear up your difficulty.
- Good question: "How can you tell the difference between the equation of a circle and the equation of a line?"
- Okay question: "How do you do #17?"
- Better question: "Can you show me how to set up #17?" (the Instructor can let you try to finish the problem on your own), or "This is how I tried to do #17. What went wrong?" The focus of attention is on your thought process.

- Right after you get help with a problem, work another similar problem by yourself.

### You Control the Help You Get

Helpers should be coaches, not crutches. They should encourage you, give you hints as you need them, and sometimes show you how to do problems. But they should not, nor be expected to, actually do the work you need to do. They are there to help you figure out how to learn math for yourself.

- When you go to office hours, your study group or a tutor, have a specific list of questions prepared in advance. You should run the session as much as possible.
- Do not allow yourself to become dependent on a tutor. The tutor cannot take the exams for you. You must take care to be the one in control of tutoring sessions.
- You must recognize that sometimes you do need some coaching to help you through, and it is up to you to seek out that coaching.

# APPENDIX C

---

## **ANSWERS**

---

22. 1  
 23.  $\pm 2$   
 24. 2  
 25. none  
 26. 2  
 27.  $-1$  and  $5$   
 34.  $(\frac{4}{15}, -\frac{1}{5})$   
 35.  $y = -\frac{2}{3}(x - 2) + 4$  and  $y = -\frac{2}{5}(x + 3) + 6$   
 36.  $y = -3(x - 1) + 5$  and  $y = \frac{1}{3}(x + 1) - 1$   
 37.  $(-\frac{27}{4}, \frac{5}{8})$   
 38.  $f(x) = 4x - 7$   
 40. 20  
 42.  $\frac{7}{2}$   
 43.  $y = \frac{1}{4}x - 1$   
 46.  $(3u - 2)(11u - 5)$   
 47. not factorable  
 48.  $(x - 9)(x - 3)$   
 54.  $(2x + 1)(2x - 1)(4x^2 + 2x + 1)(4x^2 - 2x + 1)$   
 55.  $(x + 7)(x^2 - x + 19)$   
 57.  $(p + 2)(p^2 - 5)(p^2 - 2p + 4)$   
 60.  $\frac{1}{4}(7x + 7)$   
 62.  $\frac{y^2}{y - 1}$   
 63.  $\frac{xy(x + y - 2)}{(x - 1)(y - 1)}$   
 64.  $\frac{3}{7}(21 - \sqrt{7})$   
 67.  $\frac{1}{13}(11 - 6\sqrt{3})$   
 69.  $3(\sqrt{2x + 3} + \sqrt{2x})$   
 70.  $5(\sqrt{x + 5} + \sqrt{5})$   
 71.  $\frac{2}{77}(29 - 13\sqrt{15})$   
 73. all real numbers  
 74. all real numbers  
 75. all real numbers  
 76.  $\{x|x \geq -1\}$   
 77. all real numbers  
 78.  $\{x|x \neq 0\}$   
 81. (a)  $\{x|x \neq 3\}$  (b)  $\frac{2}{3}$  (c)  $x = 3, y = 1$   
 82. (a)  $\{x|x \neq -4, 2\}$  (b) 0 (c)  $x = -4, x = 2, y = 0$   
 84. (a)  $\{x|x \neq 4\}$  (b)  $-\frac{3}{4}$  (c)  $x = 4$   
 88.  $(3z + 8)(5z + 4)$   
 91.  $(3x + y)(9x^2 - 3xy + y^2)$   
 92.  $(2w^2 + 1)(w - 5)$   
 97.  $-\frac{1}{2}$   
 99.  $-5$   
 101.  $\frac{2}{3}$   
 102. 0  
 104.  $\frac{3}{4}\sqrt{2}$   
 109.  $x = 1, y = 2$   
 110.  $x = \frac{1}{3}, y = \frac{2}{5}$   
 111.  $k = -2$   
 114. (a) 5 (b) 5 (c) 5 (d)  $-5$   
 117. (a) 3 (b)  $-3$  (c) d.n.e. (d) undefined  
 131.  $x = y = \frac{1}{3}$   
 132.  $k = -3, m = 3$   
 134. (a)  $\{x|x \neq 0, -2\}$  (b) none (c)  $x = -2$   
 135. (a)  $\{x|x \neq 0\}$  (b) none (c)  $x = 0$   
 136.  $-10$   
 137.  $-1$   
 138.  $\frac{5}{7}$   
 139. 20  
 140. 4  
 141.  $-\frac{1}{2}$   
 152. true  
 153. true  
 154. false  
 155. true  
 156. true  
 157. true  
 158. false  
 159. false  
 160. false  
 161. false  
 162. true  
 163. false  
 164.  $\frac{1}{2\sqrt{3}}$   
 165.  $-\frac{1}{4}$   
 166. 1  
 167. 0  
 168.  $e$   
 169.  $\frac{1}{e}$   
 170.  $f$   
 171.  $f$   
 172.  $-\infty$   
 173.  $\infty$   
 174. d.n.e.  
 175. 0  
 176.  $-\infty$   
 177.  $\infty$   
 178. d.n.e.  
 179. d.n.e.  
 180.  $k$   
 181.  $f$   
 182.  $i$   
 183. 0  
 184. undefined  
 185.  $k$   
 187.  $x^2 + 2x + 4$   
 188. (a) 10 (b) 30 (c) 20 (d) 40.2  
 189.  $c + 3$   
 190.  $\frac{\sqrt{c} - 1}{c - 1}$   
 191.  $2(c - 1)$   
 192. 2  
 193.  $-\frac{3}{c}$   
 194.  $-2(c^2 + c + 1)$   
 195.  $xy$

196.  $3x$   
 197.  $y + 3$   
 198.  $x + 2y$   
 199.  $5x + \ln 6$   
 200.  $\frac{x^3}{25}$   
 201.  $\{x|x > 0\}, 0$   
 203.  $\{x|x \neq \frac{1}{2}\}, \sqrt[3]{-7}$   
 204.  $\left\{x|x < -\sqrt{\frac{1}{2}}, x > \sqrt{\frac{1}{2}}\right\}$ ,  
 none  
 206.  $\{x|x < 3 \cup x > 5\}, \ln 15$   
 210. (a) 2 (c)  $y = 2x$  (d)  $2x + 1 + \Delta x$   
 211. (a) 19 (b) 1  
 212. (a)  $\frac{4}{3}$  (b) 1  
 213. (a)  $-\frac{4}{\pi}$  (b)  $-\frac{4\sqrt{3}}{\pi}$   
 214. (a)  $-\frac{2}{\pi}$  (b) 0  
 215.  $-\frac{5}{2}$   
 217.  $-\frac{3}{2}$   
 218. 0  
 219.  $\infty$   
 220.  $\infty$   
 223. 0  
 224. d.n.e.  
 227. d.n.e.  
 230. d.n.e.  
 231.  $-7$   
 233. 2  
 235.  $-2$   
 238. 0  
 239. all reals except 2  
 240. all real numbers  
 241. all reals except  $[1, 2)$   
 242. all real numbers  
 243. all reals except 1  
 244. all reals except  $\pm 1$   
 254.  $\frac{1}{5} \ln 7$   
 255. 1  
 256.  $\frac{1}{3}(\ln 12 - 7)$   
 257.  $e^2 - 1$   
 258. 0  
 259. 9  
 260. 9  
 261. 1  
 262.  $\pm \sqrt{\frac{\ln 7}{\ln 3}}$   
 263. 1  
 267.  $-3, 0, 3$   
 270.  $-1, 1, 5$   
 272. 2  
 273. none  
 275. 0  
 276. 0, 2  
 278.  $-3$   
 280.  $-1$   
 281. 0  
 282.  $0, \frac{3}{5}$   
 283. 7  
 284.  $\pm \sqrt{\frac{5}{8}}$   
 285. 0  
 291. neither  
 300. none  
 301.  $-7$   
 302. 0  
 303. 3  
 304.  $\frac{1}{2}$   
 305. 1  
 306. (b)-(g) yes (h)-(k) no (l) all reals except 0, 1, 2 (m) 0 (n) 2  
 307. no, there is a hole where  $x = -2$   
 308. no, there is a hole where  $x = -4$   
 309.  $a = 2$   
 310.  $a = 4$   
 311.  $a = -1, b = 1$   
 312.  $a = 4$   
 313. C  
 314. B  
 315. D  
 316. E  
 317. D  
 318. C  
 319. C  
 320. E  
 321. A  
 322. A  
 323. C  
 324. D  
 325. D  
 326. B  
 327. C  
 328. (a) 2 (b)  $-2$  (c)  $-\infty$  (d)  $-\infty$  (e)  $x = 0$  (f)  $y = \pm 2$   
 329. (a)  $\{x|x \neq 0\}$  (b) none (c) none (d) 0 (e) 1 (f) d.n.e.  
 330. (a) odd (b) all nonzero multiples of  $\pi$  (c) d.n.e.  
 331. (b) all reals in  $(0, 1)$  and  $(1, 2)$  (c) 2 (d) 0  
 347.  $(5x^{3/5} - 7x^{4/3})(5x^{3/5} + 7x^{4/3})$   
 348.  $2x^{-7/3}(2 - 3x^{2/3} + 6x^{4/3})$   
 349.  $x^{-3}(x+1)(x^5 - 1)$   
 350.  $\frac{2}{3}x^{4/3}(2x^{1/3} + 3)(x^{1/3} + 4)$   
 351.  $\frac{(2x+3)\sqrt{x^2+3x}}{2x(x+3)}$   
 352.  $\sqrt{x+3}$   
 353.  $x^{-2/3}(x^{1/3} + 1)(2x + 5 + 2x^{2/3})$   
 354.  $\frac{-2(x-4)}{3x^{7/3}(x-2)^{1/3}}$   
 355.  $\frac{x^2 - 7}{2x^{3/2}\sqrt{x^2 + 7}}$   
 356.  $\frac{11 - x}{2(x-3)^2\sqrt{x-7}}$   
 371. 1

- 372.**  $2x - 3 + 5x^{-2} - 14x^{-3}$   
**373.**  $12x + 13$   
**374.**  $\frac{1}{2}x^{-1/2} + \frac{34}{3}x^{-1/3}$   
**375.**  $2\pi x^2 + 20\pi x$   
**380.**  $14x, 14, 14x, 28$   
**381.** no  
**382.** no  
**383.** yes  
**384.** no  
**385.** yes  
**386.** yes  
**392.**  $(-2, -5)$   
**394.**  $6\pi$   
**396.** yes  
**397.** (a)  $8(x - 2) = y - 1$  (b)  $[-4, \infty)$  (c)  $8(x - 2) = y - 1$  and  $8(x + 2) = y - 1$   
**398.** no  
**399.** no  
**400.** yes  
**401.** yes  
**402.** no  
**403.** no  
**404.** no  
**405.** no  
**409.** (a)  $6t$  (b)  $-6$   
**410.** (b)  $-(x + 1) = y + 2$   
**412.** (a) 280 (b) mg/day  
**428.** a, d, and e  
**429.** d.n.e.  
**430.** 0  
**431.** 0  
**432.**  $\frac{\pi}{2}$   
**433.**  $-\frac{\pi}{2}$   
**434.**  $\frac{\pi}{2}$   
**435.** only one is even, only one is neither  
**436.**  $0, \frac{2\pi}{3}, \frac{4\pi}{3}$   
**437.**  $\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$   
**438.**  $\frac{5\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{15\pi}{12}, \frac{21\pi}{12}, \frac{23\pi}{12}$   
**439.**  $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$   
**440.**  $\frac{2\pi}{3}, \frac{\pi}{2}, \frac{4\pi}{3}, \frac{3\pi}{2}$   
**441.**  $\pi$   
**442.** (a) 9000 gal (b) 300 gal/hr (c) yes, the tank's volume is zero at  $t = 30$  hrs  
**444.** (a) yes (b)-(d) no  
**445.** none must be true  
**446.** (a)  $a = b + 2$  (c)  $a = 3, b = 1$   
**453.**  $-12y^2(y^3 - 5)^{-5}$   
**455.**  $\frac{-3p^4 + 21p^2 - 36p + 10}{(p^3 + 2p - 6)^2}$   
**456.**  $\frac{-3}{2x^{5/2}\sqrt{5}}$   
**458.**  $\frac{-z}{(36 - z^2)^{3/2}}$   
**460.**  $\frac{10u + 5}{6\sqrt{u - 1}(2u + 3)^{2/3}}$   
**461.**  $\frac{15}{(x + 5)^2}$   
**463.**  $\frac{-20(x + 5)}{(x - 5)^3}$   
**464.**  $\frac{7}{(1 - 3x)^2}$   
**466.**  $\frac{-24x^2 + 80x + 47}{(5 - 3x)^2}$   
**483.**  $-3(x - \frac{\pi}{4}) = y - 4; \frac{\pi}{2}$   
**485.**  $\cos x$   
**486.**  $\pi(x - 1) = y - 2$   
**487.**  $\csc \theta(\cot^2 \theta + \csc^2 \theta)$   
**488.**  $\sec \theta(\tan^2 \theta + \sec^2 \theta)$   
**489.**  $2 \sin \theta$   
**490.**  $-\sin \theta - \cos \theta$   
**497.**  $10(x - 1) = y - 2$   
**499.** no  
**500.** no  
**501.** no  
**502.** yes  
**503.** no  
**504.** no  
**505.** (a) 5 (b) 0 (c) 8 (d) 2 (e) 6 (f)  $-1$   
**517.**  $y = 1$   
**518.** (a)  $\frac{5}{4}(x - 4) = y - 2$  and  $\frac{4}{5}(x - 2) = y - 4$  (b) 0 and  $3\sqrt[3]{2}$  (c) 0 and  $3\sqrt[3]{4}$   
**519.**  $(\pm\sqrt{7}, 0)$ ; slope is  $-2$   
**520.**  $(3, -1)$   
**521.** (a)  $\frac{3x^y - y^2}{2xy - x^3}$   
 (b) at  $(1, -2)$  the tangent is  $2(x - 1) = y + 2$ , at  $(1, 3)$  the tangent is  $y = 3$   
 (c)  $\sqrt[5]{-24}$   
**530.**  $AC = \frac{5}{4}\sqrt{29}, BC = \frac{25}{2}$   
**531.**  $16\sqrt{3}$   
**532.**  $50\pi$   
**533.** 15  
**534.**  $3\sqrt[3]{9}$   
**535.**  $6\sqrt{3}$   
**536.**  $2\sqrt{3}$   
**545.** (a) 34994 dollars/week (b) 200 dollars/week (c) 34794 dollars/week  
**546.**  $\frac{5}{16}\sqrt{3}$  m/hr  
**547.** 18 m/sec  
**548.** 3 ft/sec  
**550.**  $(\frac{1}{4}, \frac{1}{2})$   
**551.** (a)  $s = \frac{7}{8}d$  (b)  $\frac{35}{8}$  ft/sec  
**552.** 1 ft/min; 40 $\pi$  ft<sup>2</sup>/min  
**553.**  $\approx 7.1$  in/min  
**554.** 12 in<sup>3</sup>/sec  
**555.** (a)  $\pm\frac{5}{3}$  units/sec (b)  $\pm 24$   
**572.**  $\ln x$   
**579.**  $\frac{2^x \ln \frac{2}{5} + \ln 5}{5^x}$   
**582.**  $\frac{3}{5(3x - 2)}$   
**584.**  $\frac{x \ln x - x + 2}{x(\ln x)^2}$



585.  $-6x^2e^{-2x^3}$
586.  $\frac{e^x(x-3)}{x^4}$
587.  $-2x^{-3}$
588.  $6(x-1)10^{3x^2-6x} \ln 10$
589.  $3^{2x}2^{3x^2}(\ln 9 + 6x \ln 2)$
590.  $\frac{2xy+y}{3xy-x}$
591.  $\frac{1}{3}(4x+12y-17)$
592.  $\frac{y}{ye^y+1}$
593.  $\frac{4\cos(x-3y)}{1+12\cos(x-3y)}$
594.  $\frac{2}{3\cos y-2}$
595.  $\frac{\sin(x-2y)}{2\sin(x-2y)-3}$
598.  $\frac{-5\csc^2 5x}{2\sqrt{\cot 5x}}$
599.  $24\cos 16x$
601.  $-6\sin 6x$
602.  $e^{\sin x} \cos x$
603.  $-3^{\cos x} \ln 3 \sin x$
604.  $\frac{2}{\ln 3} \cot 2x$
605.  $6x$
606.  $e^{3x}(\sec^2 x + 3 \tan x)$
607.  $\frac{-2e^{1/x^2}}{x^3}$
608.  $\frac{1}{2}xe^{x^2/4}$
610.  $e^{\tan x}(1+x\sec^2 x)$
625.  $\frac{2}{\sqrt{3}}$
626.  $\frac{1}{2}$
627.  $\frac{1}{51}$
628.  $\frac{1}{2}$
629.  $\frac{\sqrt[3]{25}}{2}$
631.  $\frac{-1}{\sqrt{-x^2+3x-2}}$
634.  $\frac{-3}{x^2+9}$
635.  $\frac{1}{|x|\sqrt{x^2-1}}$
636.  $\frac{-4}{\sqrt{2-4x^2}}$
637.  $\frac{-1}{\sqrt{2x-x^2}}$
638.  $y = ex$
644. (a) 6.7 million ft<sup>3</sup>/acre  
(b) 0.073 and 0.04 million ft<sup>3</sup>/acre per year
647. (b) 50 (c) 25 (d)  $1-0.04x$   
(e) 0
648. (a)  $x'(t) = \frac{1}{1+t^2}$  is always positive (b)  $x''(t) = \frac{-2t}{(1+t^2)^2}$  is always negative (c)  $\frac{\pi}{2}$
649. (a)  $\frac{10}{\sqrt{2}}$  (b) left -10, right 10 (c) when  $t = -10$ ,  $v = 0$  and  $a = 10$ , when  $t = 10$ ,  $v = 0$ ,  $a = -10$  (d) at  $t = -\frac{\pi}{4}$ ,  $v = -10$ , speed = 10,  $a = 10$
650. (a)  $2x$  (b)  $2x$  (c) 2 (d) 2  
(e) yes
651. (a)  $x = -1$  (b)  $\approx -1$
652. (a)  $\mathbb{R}$  (b)  $\frac{-\cos x}{2\sqrt{1-\sin x}}$  (c)  $\{x|x \neq \frac{\pi}{2} + 2\pi n, n \in \mathbb{Z}\}$   
(d)  $y = -\frac{1}{2}x + 1$
653. (a)  $-\frac{2x+y}{x+2y}$  (c) (6, -3) and (-6, 3)
654. (a)  $\frac{24}{\pi}$  in/sec (b)  $\frac{120}{\pi} - 30$  in<sup>2</sup>/sec
655. (a)  $\frac{3}{\sqrt{5}}$  m/sec (b) 150 m<sup>2</sup>/sec (c)  $\frac{3}{125}$  radian/sec
656. (a)  $a = 0, b = 9, c = 4$  (b)  $x = \pm 2$  (c)  $y = 0$
658. E
659. D
660. D
661. D
662. E
663. E
664. E
665. B
666. B
667. C
668. D
669. D
670. B
671. D
672. D
673. A
674. D
675. E
681. (a)-(c) no (d)  $(\pm 3, 0)$ ,  $(\pm\sqrt{3}, 6\sqrt{3})$ , and  $(0, 0)$
682.  $\{x|0 < x < 5\}$ , extreme values are 0 and 144
693.  $c = 1$
694.  $c = \sqrt{\frac{7}{3}}$
695.  $\frac{5}{6}$
696. -1, 0, 1
698.  $0, \frac{1}{5}$
699. No, Rolle's Theorem does not apply since  $f$  is not continuous on  $[0, 1]$ .
709.  $a = -\frac{19}{9}, b = \frac{11}{3}$
710.  $a = 6, b = 27, c = 36, d = 16$
711. (a) (0, 0) (b) ccup for  $x > 0$ , ccdown for  $x < 0$
714. (a)  $(\pm\frac{2}{\sqrt{3}}, \frac{9}{16})$  (b) ccup for  $x < -\frac{2}{\sqrt{3}}$  and  $x > \frac{2}{\sqrt{3}}$ , ccdown for  $-\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$
715. (a) (3, 6) (b) ccup for  $x > 3$ , ccdown for  $x < 3$
717.  $a = -3, b = -6$
718.  $a = -1, b = -3, c = -5$
722. (b) max at  $x = -2$ , min at  $x = 0$

726. (a)  $\{x|x \neq \pm 3\}$  (b) 0 (c) 0 (d) min at  $(0, 0)$  (e) inc for  $x < -3$  and  $-3 < x < 0$ , dec for  $0 < x < 3$  and  $x > 3$  (f) none (g) ccup for  $x < -3$  and  $x > 3$ , ccdow for  $-3 < x < 3$
729. (a)  $\{x|x > 0\}$  (b) 1 (c) none (d) max at  $(e, \frac{1}{e})$  (e) inc for  $0 < x < e$ , dec for  $x > e$  (f)  $(e^{3/2}, \frac{3}{2e^{3/2}})$  (g) ccup for  $x > e^{3/2}$ , ccdow for  $0 < x < e^{3/2}$
732. 3.84
733. (a) mins at  $x = -2.5$  and  $x = 2$ , max at  $x = 0$  (b) ccup for  $-3 < x < -1$  and  $1 < x < 3$ , ccdow for  $-1 < x < 1$  and  $3 < x < 4$
735. (a) at  $t = \frac{2}{3}$   $x = \frac{53}{81}$ , at  $t = -1$   $x = \frac{1}{12}$  (b) at  $t = \frac{1}{12}$   $x = -\frac{2}{3}\sqrt{3}$
736. (a) 0 (b) 6 (c) always right
737. (a)  $v(t) = -2\pi t \sin(\frac{\pi}{2}t^2)$  (b)  $a(t) = -2\pi(\sin(\frac{\pi}{2}t^2) + \pi t^2 \cos(\frac{\pi}{2}t^2))$  (c) right for  $-1 < t < 0$ , left for  $0 < t < 1$  (d) 0
738. (a)  $3\pi t - 3\pi t \cos(\frac{3\pi}{2}t^2)$  (b)  $3\pi - 2\pi \cos(\frac{3\pi}{2}t^2) + 9\pi^2 t^2 \sin(\frac{3\pi}{2}t^2)$  (c)  $0, \sqrt{\frac{4}{3}}, \sqrt{\frac{8}{3}}$  (d)  $0, 2\pi, 4\pi$
740. (a)  $4e^{3t} - 8$  (b)  $12e^{3t}$  (c)  $\frac{1}{3} \ln 2$  (d)  $\frac{8}{3}(1 - \ln 2)$
741. (a) 135 sec (b)  $\frac{5}{73}$  furlongs (c)  $\frac{1}{13}$  furl/sec (d) the last and first furlong
746. one piece 14.8 m, other 15.2 m; use all iron to make the triangle
747.  $8 \times 8 \times 4$  cm
748. 42
749.  $225 \times 150$  m
750. \$2.95
751.  $\frac{\pi}{4}$
752. (a)  $\approx 578.7$  cm<sup>3</sup> (b)  $616\frac{2}{3}$  cm<sup>2</sup>
753. (a)  $[0, B]$ , max dosage, scale factor (b)  $\frac{2}{3}B$  (c)  $\frac{4}{27}AB^3$  (d)  $\frac{1}{3}B$
754.  $R^2$
755. 10 shipments of 240 players each
756.  $\frac{1}{\sqrt{2e}}$
757. (a)  $\frac{\pi}{3}, \frac{4\pi}{3}$  (b) 1 (c)  $\frac{\pi}{3}, \frac{4\pi}{3}$
759. -1
769. crit pt is  $x = 1$ , inc for  $x < 1$ , dec for  $x > 1$ , extrema at  $x = 1$
778. (c)  $\arctan x + \frac{x}{1+x^2}$  (e)  $-25x^{-2} + 6x^{-1/2}$  (f)  $30x^4 - 60x^3 + 20x - 21$  (g)  $\frac{-2(x^2 + 1)}{(x^2 - 1)^2}$
780.  $y = \frac{1}{2}t$
781. (d)  $y = e^x$
782.  $y' = \cot x$
783.  $e^3$
788.  $\frac{\pi}{3}$
789. (a) 4, 0 (b) -1, -1, 1,  $\frac{1}{2}$  (c) 0,  $-\frac{3}{2}$
790. (a) 1,  $\frac{3}{4}$  (b) positive (c) zero
791. (a)  $h' = 0$  (b)  $k' = 0$
793.  $\frac{1}{4}$
794. (a) odd (b)  $\frac{1 + \cos x + x \sin x}{\cos^2 x}$  (c)  $y = 2x$
795. A
796. E
798. (a) max at  $x = -1$ , mins at  $x = \pm 3$  (b)  $x = 0, x = 1$
799. (b)  $x = 0$  (c) everywhere
800.  $\mathbb{R}$ , min at  $(0, \frac{1}{10})$
801.  $\mathbb{R}$ , max at  $(0, 10)$
803.  $\{x|x \neq -1\}$ , no extrema
804.  $\{x|x > 0\}$ , no extrema
806.  $e^{-x}(x - 2)$
807.  $e^x(x^2 + 4x + 2)$
808.  $e^{x+e^x}(1 + e^x)$
810. (a)  $\frac{-2xy}{x^2 + y^2}$  (b)  $y = \frac{4}{5}x - \frac{13}{5}$  (c)  $\sqrt[3]{-13}$
811. (a)  $0, \frac{\pi}{2}, \pi$  (b)  $\frac{\pi}{6} < x < \frac{\pi}{2}$  and  $\frac{5\pi}{6} < x < \frac{3\pi}{2}$  (c) min of  $-\frac{1}{4}$ , max of 2
812. (a)  $y = 4x + 2$  and  $y = 4x - 2$  (b) 1 (c) 0
813. (a)  $x = -2$  (b)  $x = 4$  (c)  $-1 < x < 1$  and  $3 < x < 5$
814. (a)  $\{x|x \neq 0\}$  (b) even (c) maxs at  $x = \pm 1$  (d)  $f(x) \leq \ln \frac{1}{2}$
815. (b)  $c \approx 1.579$  (c)  $y \approx 1.457x - 1.075$  (d)  $y \approx 1.457x - 1.579$
817. (a)  $k = -2, p = 2$  (b) always inc (c) (1, 1)
818. (a) min of  $\frac{-e^{5\pi/4}}{\sqrt{2}}$ , max of  $e^{2\pi}$  (b) inc for  $0 < x < \frac{\pi}{4}$  and  $\frac{5\pi}{4} < x < 2\pi$  (c)  $\pi$
819. (a) 100 (b)  $y = \frac{3}{5}x + 20$  (c) yes, the top 5 ft of the tree
820. C
821. B
822. B
823. A
824. D
825. C
826. D
827. C
828. D
829. E
830. B
831. E

- 832.** A  
**833.** B  
**834.** B  
**857.**  $\frac{1}{4}x^4 + 2x + C$   
**858.**  $\frac{1}{3}x^3 - x^2 + 3x + C$   
**859.**  $\frac{2}{5}x^{5/2} + x^2 + x + C$   
**860.**  $\frac{2}{3}x^{3/2} + x^{1/2} + C$   
**861.**  $\frac{3}{5}x^{5/3} + C$   
**862.**  $-\frac{1}{2}x^{-2} + C$   
**863.**  $x - \frac{1}{x} + C$   
**864.**  $\frac{2}{7}x^{7/2} + C$   
**865.**  $3x + C$   
**866.**  $\frac{1}{3}x^3 + \cos x + C$   
**867.**  $x + \csc x + C$   
**868.**  $\tan \theta + \cos \theta + C$   
**869.**  $\sec \theta - \tan \theta + C$   
**870.**  $20x^{2/5} + C$   
**871.**  $-\frac{9}{2}x^{2/3} + C$   
**872.**  $\frac{21}{8}x^8 - \frac{14}{5}x^5 + C$   
**873.**  $\frac{7}{4}x - \frac{3}{10}x^{5/2} - \frac{3}{2}x^{1/2} + C$   
**895.**  $\frac{8}{3^{7/5}}(5x - 2)^{3/2}(15x + 4) + C$   
**896.**  $-\cos(4x^3) + C$   
**897.**  $\sin(4e^x) + C$   
**898.**  $3^{3t-1} + C$   
**899.**  $\frac{1}{4}(6^{2z^2-3}) + C$   
**900.**  $\frac{1}{\ln 32}(2^{5x}) + C$   
**901.**  $\frac{2}{5}\sqrt{5x+4} + C$   
**902.**  $-\frac{1}{3}(7-3y^2)^{3/2} + C$   
**903.**  $\frac{1}{3}\sin(3z+4) + C$   
**904.**  $-e^{1/t} + C$   
**905.**  $\sec(x + \frac{\pi}{2}) + C$   
**906.**  $\frac{2}{3}(\cot \theta)^{3/2} + C$   
**907.**  $\frac{1}{2}\ln|x^2+4| + C$   
**908.**  $\frac{1}{2}\arcsin(2x) + C$   
**909.**  $\arctan(e^x) + C$   
**919.** (a)  $v(t) = 18t - t^2 + 19$ ,  
 $a(t) = 9t^2 - \frac{1}{3}t^3 + 19t - \frac{20}{3}$   
(b) 1317 m  
**920.** (a) 87 (b) 87  
**921.** (a) 0.969 mi (b) 22.7 sec,  
120 mph  
**922.** (a) 758 gal, 543 gal (b)  
2363 gal, 1693 gal (c) 31.4  
hrs, 32.4 hrs  
**923.** 799500 ft<sup>3</sup>  
**936.**  $-\frac{4}{3}\sqrt{2} + 1$   
**943.** -3  
**947.**  $\frac{51}{2}$   
**951.** 7  
**953.**  $\pi$   
**955.** (a) 2 (b) negative (c)  $\frac{9}{2}$   
(d) 6 (e) 4 and 7 (f) to  
for  $6 < t < 9$ , away for  
 $0 < t < 6$  (g) the right  
side  
**960.**  $-\frac{1}{8}(4x^2 - 1)^{-4} + C$   
**961.**  $\frac{1}{2}z^2 - 2z + \frac{5}{z} + C$   
**962.**  $\frac{1}{71}(x+7)^{71} + C$   
**963.**  $\frac{1}{8}(e^x - 1)^8 + C$   
**970.**  $\frac{1}{2}$   
**972.**  $\frac{4}{15}$   
**977.**  $\frac{4752}{35}$   
**979.** 2  
**993.**  $\ln|\sec \theta - 1| + C$   
**994.**  $e^{5x} + C$   
**995.**  $-\ln(1 + e^{-x}) + C$   
**996.**  $-\frac{2}{3}(1 - e^x)^{3/2} + C$   
**997.**  $\ln|e^x - e^{-x}| + C$   
**998.**  $-\frac{5}{2}e^{-2x} + e^{-x} + C$   
**999.**  $\frac{1}{\pi}e^{\sin(\pi x)} + C$   
**1000.**  $\ln|\cos(e^{-x})| + C$   
**1001.**  $\frac{1}{\ln 3}3^x + C$   
**1002.**  $\frac{-1}{\ln 25}5^{-x^2} + C$   
**1003.**  $\frac{1}{\ln 9}\ln|1 + 3^{2x}| + C$   
**1006.**  $\ln 2$   
**1009.**  $4 + 5 \ln 5$   
**1011.**  $\frac{1}{3}e(e^2 - 1)$   
**1019.** (a)  $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$   
**1020.** (a)  $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}$   
**1022.** (b) -2  
**1023.** (a) 0 (b) no, yes (c)  $\sqrt{12}$   
(d)  $4\pi, -4\pi$   
**1026.** 0  
**1028.**  $2xf(x^2)$   
**1037.** 16  
**1039.** 86  
**1040.**  $\ln 4 + \frac{21}{2}$   
**1041.**  $\frac{68}{3}$   
**1042.**  $\frac{271}{6}$   
**1046.** (a)  $\frac{77}{16}$  (b)  $\frac{85}{16}$   
**1047.** (b) 66 degrees  
**1048.** left is 10, right is 7.25  
**1051.** (a) left is 4.06, right is  
4.77 (b) 4.36  
**1058.** (a)  $\frac{107}{16}$  (b)  $\frac{51}{16}$  (c)  $\frac{79}{16}$  (d)  
 $\frac{19}{4}$  (e)  $\frac{19}{4}$   
**1061.**  $\frac{5}{2}$   
**1062.**  $\ln \sqrt{2}$   
**1063.**  $\pi$   
**1066.**  $\frac{2^{n+1} - 1}{n + 1}$   
**1071.** net is  $-\frac{9}{2}$ , total is  $\frac{29}{6}$   
**1072.** net is 0, total is  $4e - \frac{4}{e}$   
**1073.** (a) 0 (b) positive (c) -9  
(d)  $t = 6$  (e)  $t = 7$  (f)  
to from  $3 < t < 6$ , away  
from  $0 < t < 3$  and  $6 <$   
 $t < 10$  (g) the right side  
**1074.** (a)-(d) true (e) false (f)  
false (g) true  
**1082.** (a) 9920 (b)  $10413\frac{1}{3}$   
**1084.** 4.2 liters  
**1085.** 2.42 gal, 24.83 mpg  
**1087.**  $\frac{45\pi}{2}$   
**1090.** (a) 750 ft (b) 550 ft (c)  
-32, 0

- 1091.** (a) 0 (b)  $-1$  (c)  $-\pi$  (d) 1  
 (e)  $y = 2x + 2 - \pi$  (f)  $-1$   
 and 2 (g)  $[-2\pi, 0]$
- 1092.** (a) 63 (b) 234.9
- 1093.**  $-\cos x + \sin x + 2$
- 1094.** (a)  $f(x) = x^3 + 4x^2 + 3x - 2$  (b)  $-\frac{2}{3}$
- 1095.** (a) up (b)  $-2.049$ , no (c) 3.827 (d) 1.173
- 1096.** (a)  $A$  (b)  $\frac{1}{2}A$  (c) 4
- 1097.** (a) 0.316
- 1098.** (a)  $x(t) = 4t^3 - 18t^2 + 15t - 1$  (b)  $\frac{1}{2}, \frac{5}{2}$  (c) max of 15 (d) 17
- 1099.** (a)  $-23$  (b) 33 (c) 11, 16,  $-8$  (d)  $a, c$
- 1100.** (a) 3 (c)  $f(x) = \frac{3}{x}$
- 1101.** (a) 258.6 gal (b) yes (c) 10.785 gal/hr
- 1102.** D
- 1103.** C
- 1104.** D
- 1105.** E
- 1106.** E
- 1107.** B
- 1108.** B
- 1109.** D
- 1110.** A
- 1111.** D
- 1112.** C
- 1113.** B
- 1114.** C
- 1115.** B
- 1116.** D
- 1117.** C
- 1118.** E
- 1119.** E
- 1125.**  $\frac{\pi}{8}(\ln 16 - \frac{5}{2})$
- 1126.**  $\frac{\pi}{4}(\ln 16 - \frac{5}{2})$
- 1128.**  $\frac{512\sqrt{2}}{15}$
- 1129.** (a)  $2\sqrt{3}$  (b) 8
- 1131.**  $\frac{8}{5}\sqrt{3}$
- 1133.**  $\pi$
- 1134.**  $\frac{\pi}{2}$
- 1135.**  $8\pi$
- 1136.**  $\frac{\pi(e^6 - 1)}{3e^9}$
- 1137.**  $4\pi$
- 1141.**  $\frac{160625\pi}{14}$
- 1142.** (a)  $\frac{459\pi}{10}$  (b)  $\frac{405\pi}{7}$  (c)  $\frac{243\pi}{20}$
- 1144.**  $\frac{\pi}{2}(5e^6 + 1)$
- 1145.**  $\pi((e - 1)\ln 16 - \frac{1}{2e^2})$
- 1146.**  $12\pi$
- 1153.**  $\frac{1}{27}(31^{3/2} - 8)$
- 1154.**  $\frac{2}{3}(10^{3/2} - 5^{3/2})$
- 1157.**  $3\sqrt{3}$
- 1158.**  $\frac{3\pi}{2}$
- 1159.**  $\frac{576\pi}{7}$
- 1160.**  $\frac{\pi^2}{2}$
- 1162.**  $9\pi(\frac{1}{e} - \frac{1}{3})$
- 1163.**  $\frac{\pi^2}{3}$
- 1164.**  $\frac{2\pi}{15}$
- 1177.**  $u = \frac{-1}{\sqrt[3]{3(v^2 + C)}}$
- 1178.**  $y = \frac{4}{3}x^{3/2} + C$
- 1179.**  $y = \frac{1}{6}(3x + 5)^4 + C$
- 1180.**  $s = \sin t - \cos t$
- 1181.**  $r = \cos(\pi\theta) - 1$
- 1182.**  $v = 3\text{arcsec } t - \pi$
- 1183.**  $v = 1 + 8\arctan t + \tan t$
- 1184.**  $y = x^2 - x^3 + 4x + 1$
- 1185.**  $y = \frac{1}{x} + 2x - 2$
- 1191.**  $P = \frac{-P_0}{P_0kt - 1}$
- 1192.**  $a = \frac{1}{2}, b = -\frac{3}{2}$
- 1193.** A
- 1194.** 16 ft/sec<sup>2</sup>
- 1195.** (a)  $3000(30)^{t/2}$   
 (b) 2,700,000 (c)  $\approx 1.76$  hrs
- 1196.** 112500, 2172
- 1197.** \$81000, 2018
- 1198.** (a)  $y = 1000000(2^{-t/6})$   
 (b) 69300 gal/yr (c)  $\approx 26$  yrs
- 1199.** (a)  $T(t) = T_a + (T_0 - T_a)e^{-kt}$  (b)  $T_a$  (c) 5:12 PM
- 1204.** (a)  $\frac{e^{-x^2/2}}{\sqrt{2\pi}}$
- 1205.**  $h(t) = \frac{3}{4}t^2 + 6t + 12$ , 75 cm
- 1207.** (a) 500 (b) 1000
- 1209.**  $y(t) = c + (y_0 - c)e^{-ktA/V}$
- 1210.** 3.458
- 1211.** 9.822
- 1212.**  $-0.227$
- 1213.**  $-20.954$
- 1214.** (a)  $-4$  (b)  $\frac{1024\pi}{105}$  (c)  $-\frac{25}{4}$
- 1215.** (a) II (b) III (c) I (d) IV
- 1216.** (a) 2 (b)  $\frac{25}{4}$  (c)  $\ln \frac{9}{4}$
- 1217.** 2.109
- 1218.** (a)  $(-0.767, 0.588)$ ,  
 (2, 4), (4, 16)
- 1219.** (a) 6.234 (b) 6.236 (c) 6.238
- 1220.**  $y = (\ln x)^4$
- 1221.** (a) 1.168
- 1222.** (a)  $P(t) = 800 - 300e^{-kt}$   
 (b)  $\frac{1}{2}\ln 3$  (c) 800
- 1223.** (a)  $(-3, 1)$  (b)  $\frac{4}{3}$  (c)  $\frac{8\pi}{3}$
- 1224.** (a)  $\frac{44\pi}{3}$   
 (b)  $2\pi \int_3^5 x\sqrt{(x+1)^2 - 9} dx$
- 1225.** (a)  $2\sqrt{4x^2 + 2x}$   
 (b)  $\{x|x > 0\}$  (c) 0 (d) 7

1226. (a)  $\frac{20}{9}$   
 (b)  $\pi \int_0^2 (6x + 4 - 4x^2) dx$
1227. (a)  $\frac{\pi}{2}(e^4 + e^{-4} - 2)$   
 (b)  $2\pi(e^2 + 3e^{-2})$
1228. A
1229. E
1230. C
1231. C
1232. A
1233. C
1234. D
1235. D
1236. C
1237. E
1238. A
1239. E
1240. D
1241. E
1242. C
1253.  $y = \left(\frac{x^2}{4} - \frac{x}{8} + \frac{1}{32}\right) e^{4x} + C$
1254.  $y = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$
1255.  $y = -2\sqrt{\theta} \cos \sqrt{\theta} + 2 \sin \sqrt{\theta} + C$
1256.  $y = \theta \sec \theta - \ln |\sec \theta + \tan \theta| + C$
1257.  $e^2$
1258.  $\ln 27 - 3$
1259.  $3 - e$
1260.  $\ln \frac{1}{4} + 5e^2 - 5e - 14$
1262.  $\frac{e^{2\pi} - 1}{2\pi e^{2\pi}}$
1263. (b)  $-3\pi$  (c)  $5\pi$
1271.  $-\frac{2}{25} \left( \ln \left| \frac{x}{x-5} \right| - \frac{5}{x} \right) + C$
1272.  $\frac{1}{125} \left( \ln \left| \frac{x}{x-5} \right| - \frac{5}{x} - \frac{5}{x-5} \right) + C$
1274.  $y = \frac{2}{2 - e^{e^x - 1}}$
1275.  $y = \frac{1}{\cos \theta + 1} - 1$
1276.  $y = \ln |x-2| - \ln |x-1| + \ln 2$
1277.  $|y+1| = \left| \frac{6t}{t+2} \right|$
1280.  $p(t) = \frac{1000e^{4t}}{499 + e^{4t}}, 1.55$  days
1287.  $y = \sqrt{x^2 - 4} - 2 \operatorname{arcsec} \left( \frac{x}{2} \right)$
1288.  $y = \ln \left| \frac{x + \sqrt{x^2 - 9}}{3} \right|$
1289.  $y = \frac{3}{2} \arctan \left( \frac{x}{2} \right) - \frac{3\pi}{8}$
1290.  $y = \frac{x}{\sqrt{x^2 + 1}} + 1$
1291.  $\frac{3\pi}{4}$
1292.  $\pi \left( \frac{\pi}{2} + 1 \right)$
1294. (a)  $(e-2)\pi$  (c)  $2\pi(\ln 2 - 1)^2$
1296.  $\frac{4}{3}$
1297. (a) 0.968 (b) 7.199
1298. (a)  $7B$  (b)  $\frac{3}{4}$
1309.  $\frac{3}{2}$
1311. 1
1312. d.n.e.
1314. 1
1315.  $\sqrt{e}$
1316. 1
1317. 1
1318.  $e$
1319.  $e$
1320. 1
1321. 1
1322. 0
1323.  $\ln 2$
1324. 1
1325. -1
1326. 1
1327.  $\frac{27}{10}$
1344. -1
1347. divergent
1348. 1
1349.  $-\frac{1}{4}$
1351.  $\ln 4$
1352.  $\pi$
1353.  $2\pi^2$
1354.  $\ln 2$
1359. 0.5 ml
1360. A
1361. (b) negative (c)  $-\frac{1}{2} \ln 3$
1362. B
1364. only one is divergent
1365. only one is false
1366. (b)  $-3.75(x-1.4) = y - 2.6$  (d) 2.2
1367. (a) 1 (b) 2, 6, 24
1372. (a)  $-\frac{13}{6}$  (b) use the Intermediate Value Theorem (c)  $y = \frac{21}{2} - 3x$  (d)  $\frac{21}{6}$
1373. (a) 1 (b)  $\frac{\pi}{2}$  (c)  $\pi$
1374.  $\frac{3}{4}$
1375. 0.726
1376. (a)  $x(t) = a - \frac{a}{akt + 1}$   
 (b)  $x = \frac{a(1 - e^{(a-b)kt})}{b - ae^{(a-b)kt}}$
1377.  $a = \frac{1}{2}$ , converges to  $-\frac{1}{4} \ln 2$
1378. (a)  $\pi$  (b)  $\pi(2e-5)$  (c)  $\frac{\pi}{2}(e^2-3)$  (d)  $\frac{\pi}{2}(e^2-4e+5)$
1379. (b)  $\frac{8\pi}{27}(9(\ln 2)^2 - 6 \ln 2 + 2)$
1380. (a)  $x = \frac{n(e^{(n+1)kt} - 1)}{n + e^{(n+1)kt}}$
1381. (a)  $\pi$
1382.  $100\pi$
1384. (a) 1 (b)  $2\pi(1 - e^{-k} - ke^{-k})$  (c)  $\frac{1}{2}(1 - e^{-2k})$
1386. C
1387. C
1388. B
1389. D
1390. C

- 1391.** E  
**1392.** E  
**1393.** E  
**1394.** A  
**1395.** A  
**1396.** D  
**1397.** B  
**1398.** A  
**1399.** C  
**1400.** A  
**1401.** C  
**1402.** D  
**1403.** D  
**1410.** 0  
**1411.** 0  
**1412.** 1  
**1413.** 1  
**1414.** 0  
**1415.** divergent  
**1416.**  $e^{2/3}$   
**1417.** 1  
**1418.**  $\frac{\pi}{2}$   
**1419.** 1  
**1420.** unbounded, divergent  
**1421.** bounded, convergent  
**1426.**  $\frac{1}{3}$   
**1427.** divergent  
**1428.** divergent  
**1429.** divergent  
**1430.** 0  
**1431.**  $e$   
**1432.** 0  
**1433.** divergent  
**1434.** divergent  
**1435.** 0  
**1436.** 5  
**1437.** 0  
**1438.** increasing  
**1439.** increasing  
**1440.** decreasing  
**1441.** increasing  
**1443.** oscillating  
**1447.** 1  
**1448.** three are false  
**1458.**  $-1 < x < 1$   
**1459.**  $-1 < x < 1$   
**1463.**  $x < -1$  and  $x > 1$   
**1464.**  $1 < x < 5$   
**1465.**  $\frac{1}{e} < x < e$   
**1466.**  $\frac{23}{99}$   
**1469.**  $\frac{140}{99}$   
**1470.**  $\frac{41333}{33300}$   
**1471.**  $\frac{22}{7}$   
**1472.** 28  
**1473.** 8  
**1477.** divergent  
**1478.** divergent  
**1479.** divergent  
**1485.** divergent  
**1486.** divergent  
**1487.** divergent  
**1488.** convergent  
**1489.** divergent  
**1490.** convergent  
**1491.** divergent  
**1492.** convergent  
**1493.** convergent  
**1494.**  $a = 1$   
**1501.**  $\frac{5}{6}$   
**1502.** converges absolutely  
**1503.** converges absolutely  
**1504.** diverges  
**1505.** converges conditionally  
**1506.** converges conditionally  
**1507.** converges absolutely  
**1508.** converges absolutely  
**1509.** diverges  
**1510.** converges conditionally  
**1511.** diverges  
**1513.** 0.2  
**1514.** 0.00001  
**1529.**  $-1 \leq x \leq 1$   
**1530.**  $-1 \leq x \leq 1$   
**1531.**  $-\frac{1}{4} \leq x < \frac{1}{4}$   
**1532.**  $2 < x \leq 4$   
**1533.**  $-2 < x \leq 8$   
**1534.**  $-\frac{1}{e} \leq x < \frac{1}{e}$   
**1535.**  $x = 0$   
**1537.**  $-1 < x < 3, \frac{4}{4-(x-1)^2}$   
**1538.**  $\frac{1}{e} < x < e, \frac{1}{1 - \ln x}$   
**1546.** (b)  $\sum (-1)^{n+1} (\frac{1}{3})^n (x - 3)^{n-1}$  (c)  $0 < x < 6$   
**1547.** (b)  $\sum \frac{(-1)^{n+1} (x - \frac{\pi}{2})^{2n-1}}{(2n-1)!}$  (c)  $\mathbb{R}$   
**1550.** (b)  $\sum \frac{(-1)^n \pi^{2n} (x - \frac{1}{2})^{2n}}{(2n)!}$  (c)  $\mathbb{R}$   
**1552.** (b)  $\sum \frac{e^2 (x-2)^n}{n!}$  (c)  $\mathbb{R}$   
**1555.**  $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \dots$   
**1557.** divergent  
**1558.**  $9 + 5x - 2x^2 + 6x^3$   
**1559.**  $-1 + \frac{1}{2}(3x - \pi)^2$   
**1560.**  $1 - \frac{1}{8}(4x - \pi)^2$   
**1561.** (a)  $-1 < x < 1$  (b)  $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \dots$   
**1572.** first and last; second and third  
**1573.**  $\mathbf{a} = \mathbf{0}$   
**1574.**  $\mathbf{G}(0) = \mathbf{i}, \mathbf{G}(\frac{\pi}{3}) = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$   
**1575.**  $\|\mathbf{F}\| = |t|\sqrt{t^2 + 4}, \|\mathbf{G}\| = 1$

1578. yes, for  $t = 0$
1585. (a)  $\langle 2t, 9t^2 \rangle$  (b)  $\langle 2, 18t \rangle$   
(c)  $\langle 2, 9 \rangle$  (d)  $\sqrt{85}$
1588. (a)  $\langle \pi \cos \pi t, \pi \sin \pi t \rangle$  (b)  
 $\langle -\pi^2 \sin \pi t, \pi^2 \cos \pi t \rangle$  (c)  
 $\langle -\pi, 0 \rangle$  (d)  $\pi$
1591.  $-3\mathbf{i} + (4\sqrt{2} - 2)\mathbf{j}$
1593.  $\mathbf{R}(t) = [(t+1)^{3/2} - 1]\mathbf{i} - (e^{-t} - 1)\mathbf{j}$
1594.  $\mathbf{R}(t) = (8t+100)\mathbf{i} + (8t - 16t^2)\mathbf{j}$
1595.  $3\sqrt{13}$
1596.  $3\pi$
1597.  $\frac{1}{27}(85^{3/2} - 72^{3/2})$
1598.  $4\sqrt{13}$
1599. (a)  $(-3 \sin t)\mathbf{i} + (2 \sin t)\mathbf{j}$   
(b)  $(-3 \cos t)\mathbf{i} - (2 \sin t)\mathbf{j}$   
(c)  $-\frac{2}{3}$  (d)  $\frac{x^2}{9} + \frac{y^2}{4} = 1$  (e)  
 $-\frac{2}{3}(x - \frac{3}{2}\sqrt{2}) = y - \sqrt{2}$
1600. (a)  $\mathbf{R}'(t) = \mathbf{i} + 18t\mathbf{j}$ ,  
 $\mathbf{R}''(t) = 18\mathbf{j}$  (c)  $y = 18x - 13$
1601. (a)  $\frac{-1}{3(1 + \frac{1}{2}\sqrt{3})}$   
(b)  $(2\sqrt{3} - 4)(x - \pi) = y - 2\sqrt{3}$
1605. (a)  $\left\langle 6te^{3t^2}, \frac{2+8t^2}{t+2t^3} \right\rangle$   
(b)  $\langle 6e^2, \frac{10}{3} \rangle$  (c) no (d)  
 $\langle e^2, \ln 3 \rangle$
1606. (a)  $(\cos t)\mathbf{i} - (2 \sin 2t)\mathbf{j}$  (b)  
 $\frac{\pi}{2}, \frac{3\pi}{2}, y = 1 - 2x^2, -1 \leq x \leq 1$
1608. (a) 160 sec (b) 225 m (c)  
 $\frac{15}{4}$  m/sec (d) 80 sec
1609. (a)  $t = 2$
1628.  $\frac{\pi}{3}, \frac{5\pi}{3}$
1629.  $\frac{\pi}{2}, \frac{3\pi}{2}$
1631.  $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$
1635.  $r = 4 \cos \theta \csc^2 \theta$
1636.  $r = \sec \theta$
1637.  $r = e^\theta$
1641. (a)  $y = -\frac{1}{4}, y = 2,$   
 $x = 0, x = \pm \frac{3\sqrt{3}}{4}$  (b)  
 $y = \pm 0.267, y = \pm 4.343,$   
 $x = -1, x = \frac{9}{16}, x = -7$
1642.  $18\pi$
1643.  $\frac{\pi}{8}$
1644. 1
1645.  $\frac{\pi}{2}$
1646.  $5\pi - 8$
1647.  $\pi + 1 - \sqrt{3}$
1648.  $12\pi - 9\sqrt{3}$
1649. 8
1650.  $\sqrt{2} + \ln(1 + \sqrt{2})$
1651.  $2\pi\sqrt{2}$
1652.  $8\pi$
1653. (b)  $y = 10 - 10x$  (c) 9.236
1654. (b)  $\frac{2}{e^{2t} - 1}, y = \frac{2x}{e + 1} + 2 \ln(e + 1) - 2$  (d)  $y = 2 \ln x$
1655. (a)  $1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots + \frac{x^n}{(n+1)!}$
1656. (a)  $\frac{9\pi}{2}$  (c)  $\frac{\pi}{4}$
1657.  $-\frac{1}{2} \leq x < \frac{1}{2}$
1658. (b) divergent
1659. (a)  $4 - 4t^2 + 4t^4 - 4t^6,$   $(-1)^n(4t^{2n})$  (b)  
 $4x - \frac{4}{3}x^3 + \frac{4}{5}x^5 - \frac{4}{7}x^7,$   
 $(-1)^n \frac{(4x^{2n+1})}{(2n+1)}$  (c)  $-1 < t < 1$  (d)  $\pm 1$
1660. (a)  $\sqrt{2t+1} - 5$   
(b)  $\frac{3(\sqrt{2t+1} - 5)^2 - 3}{\sqrt{2t+1}}$   
(c)  $(-2, -2), 3.018$
1661. (b)  $8 - \pi$
1663. (a) 3.69 (b) 2.645 (c)  
 $-1.52$
1664. (a)  $-3 < x < 3$  (b)  $\frac{2}{9}$  (c)  
 $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^{n+1}}$  (d)  
 $\frac{1}{2}$
1665. (a)  $\langle 1, 2 \rangle, \langle \frac{3}{2}, \frac{3}{2} \rangle$  (c)  $t = 4$
1666. (a)  $(0, 5)$  (c) 3 times (d)  
 $\langle -3\pi \sin \pi t, 5\pi \cos \pi t \rangle$  (e)  
5.392
1667. C
1668. E
1669. C
1670. C
1671. A
1672. C
1673. D
1674. C
1675. D
1676. B
1677. E
1678. B
1679. C
1706. surface area is  $\frac{\pi}{27}$ , need  
0.25 cubic inches of glass
1715.  $y = \frac{3}{2} - \frac{1}{2}e^{-2x}$
1716.  $y = \frac{1}{x} \cot x + \frac{\pi}{2x}$
1717.  $y = 1 - 7e^{-x^2/2}$
1719. (a) 10 lbs/min (b)  $100 + t$   
gal (c)  $\frac{4y}{100+t}$  lbs/min (d)  
 $y = 100 + t - \frac{150}{(1 + \frac{t}{100})^4}$   
(e)  $\approx 1.5$  lbs/gal
1733.  $x_0 > 0 \rightarrow \sqrt{2}, x_0 < 0 \rightarrow -\sqrt{2}$
1736.  $y = \pm \sqrt{9 - x^2}$
1737.  $y = \pm \sqrt{\frac{4}{9}x^2 - 4}$
1738.  $y = -1 \pm \sqrt{4 - \frac{1}{2}(x-1)^2}$
1739.  $y = e^{x^2+5}$
1740.  $y = \ln|x+7|$
1741.  $y = \frac{x}{1-x}$
1742. 0, 2
1743. -5
1744. 0, -2
1745. 2, 6
1746.  $|x^2 - 1|, x^2 - 1$
1747.  $x, x$
1748.  $(x-1)^2 + 1, x^2$
1749.  $x+1, \sqrt{x^2+1}$

- 1750.** 1  
**1751.** 0  
**1752.** 1  
**1753.** 0  
**1754.** -1  
**1755.** 1  
**1756.** 2  
**1757.** -2  
**1758.**  $e^x(2x^2 + 4x + 3)$   
**1759.**  $5^x 2^{x-1} \ln 10$   
**1760.**  $\frac{16}{x+1} + \frac{8(4x+1)}{x(2x+1)} - \frac{x}{x^2+4}$   
**1761.**  $\frac{14e^{2x}}{2^{2x}+6} + \frac{1}{2(x+4)} + \frac{5(e^{-x}-e^x)}{e^{-x}+e^x}$   
**1762.**  $\frac{7}{(2-3x)(x+4)}$   
**1763.**  $\frac{1-\ln x}{3x^2 \ln 10}$   
**1764.**  $\sec(3x)(4 + (4x-1)\tan(3x))$   
**1765.**  $\frac{3 \sec^2 3x \tan 2x - 2 \tan 3x \sec^2 2x}{\tan^2 2x}$   
**1766.**  $\frac{(\ln 10)(\cos 2x - 2x \sin 2x \ln x)}{x(\ln x)^2}$   
**1767.**  $1 + \cos^2 x - x \sin 2x$   
**1768.**  $2x(\ln 3)(3^{x^2})$   
**1769.**  $\frac{-x^2 - 2x + 1}{(\ln 10)(x+1)(x^2+1)}$   
**1770.**  $2e^{-x}(1-x)$   
**1771.**  $\frac{6xy^2 + 3y^2 - 2xy \ln y}{2x^2 - 3xy \ln x - 6x^2 y}$   
**1772.**  $\frac{2x}{3x^2 e^{3y} - 12ye^{3y} + 4}$   
**1773.**  $\frac{1 - xe^{5y} + 2x^2 y}{x^2(5e^{5y} - x)}$   
**1774.** (a)  $2 < x < 6, 8 < x < 10$   
 (b)  $0 < x < 2, 6 < x < 8$   
 (c) 2, 8 (d) 6 (e)  $0 < x < 3, 7 < x < 9$  (f)  $3 < x < 7, 9 < x < 10$   
 (g) 3, 7, 9 (h) -6, 0  
**1775.** (a) 30 (b)  $-\frac{9}{4}$  (c) 24 (d) 4 (e) -12  
**1776.**  $-\frac{5}{3}$   
**1777.** -1, -3  
**1778.** 12, -4  
**1779.** (b)  $-5 < x < -4.715$  and  $-1.496 < x < 0.769$  (c)  $-5 < x < -3.127$  and  $-0.26 < x < 2$  (d) yes, at  $x = 0.769$  and  $x = -4.715$  (e)  $x = -3.127$  and  $x = -0.26$   
**1780.** (a)  $v(t) = 4 - 6t - 3t^2$  (b)  $a(t) = -6 - 6t$   
**1781.**  $\sqrt{2 - \sqrt{2}}$ , collide at  $t = \frac{7\pi}{8}$   
**1782.** (a) 4% (b) 8% (c) 12%  
**1783.** (a)  $\mathbb{R}$ , neither (b)  $a'(x)$  is -2 for  $x < 1$ , 0 for  $1 < x < 3$ , and 2 for  $x > 3$  (c) min is 2  
**1784.** 120  
**1785.** (a) 2000, -187 (b) 6  
**1786.** (b)  $y' = \frac{-2x-y}{x+2y}$ ,  $y' = \frac{-2x+y}{2y-x}$  (c)  $-\frac{1}{2}$  and -2,  $\frac{1}{2}$  and 2 (d) let  $z = \sqrt{\frac{1}{3}}$ :  $(z, -2z)$ ,  $(-z, 2z)$ ,  $(z, 2z)$ ,  $(-z, -2z)$  (f)  $y = x$ ,  $y = -x$   
**1790.** (a)  $\frac{1}{1+x^2}$  (b)  $\frac{\pi}{2}$ ,  $-\frac{\pi}{2}$ , 0, 0 (c) odd (d)  $f'(x) > 0$  for all  $x$  (e) ccup for  $x > 0$ , ccdown for  $x < 0$ , inf pt at the origin  
**1792.**  $\frac{2}{3} \ln(3y^2 + 2) + C$   
**1793.**  $2z^{3/2} - \frac{8}{5}z^{5/2} + 2z^{1/2} + C$   
**1794.**  $\frac{3^{5y}}{5 \ln 3} + C$   
**1795.**  $-\frac{1}{7} \cos 7x + C$   
**1796.**  $\frac{2}{15}(3x^2 - 5)^{5/4} + C$   
**1797.**  $\frac{1}{2}x - \frac{1}{16} \sin 8x + C$   
**1798.**  $\frac{1}{2}e^{\sin 2x} + C$   
**1799.**  $-\frac{1}{12} \cos^3 4x + C$   
**1800.**  $-\sin x + 2 \cos x + C$   
**1801.**  $-\frac{3}{2}x^{-2} - \ln|x| - \frac{1}{3}x^{-3} + C$   
**1802.**  $\ln|x| + C$   
**1803.**  $\frac{1}{2} \ln|2x-3| + C$   
**1804.**  $\frac{1}{4} \sin(4x-5) + C$   
**1805.**  $\frac{1}{90}(3x^2 - 2)^5 + C$   
**1806.**  $\frac{2^{3y^2}}{6 \ln 2} + C$   
**1807.**  $\ln| \sin x - 3| + C$   
**1808.**  $\frac{1}{2} \ln|\sec 2x| + C$   
**1809.**  $-\frac{1}{5}e^{1/x} + C$   
**1810.**  $\frac{1}{2} \ln|e^{2x} - 7| + C$   
**1811.**  $\frac{1}{9}x^3 - \frac{2}{25}x^5 - \frac{3}{7}x + C$   
**1812.**  $\frac{1}{12}(e^{2x} + 3)^6 + C$   
**1813.**  $\frac{6}{11}x^{11/6} - \frac{10}{19}x^{19/10} + C$   
**1814.**  $\frac{1}{4}x^4 + C$   
**1815.**  $\frac{1}{5}e^{5 \sin x} + C$   
**1816.**  $\frac{15}{11}x^{11/5} - \frac{5}{3}x^{6/5} + C$   
**1817.**  $\frac{1}{4}(x^2 + 3x - 2) + C$   
**1818.**  $\frac{1}{3} \ln|x^3 - 2| + C$   
**1819.**  $\frac{1}{6}(x^4 - 2)^{3/2} + C$   
**1820.**  $\frac{2}{5}x^{5/2} + 2x^{3/2} + C$   
**1821.**  $\frac{1}{4}y^{4/3} - \frac{9}{2}y^{2/3} + C$   
**1822.**  $\frac{4}{3}x^{3/2} + 6\sqrt{x} + C$   
**1823.**  $\frac{3}{2}(x^2 + 1)^{5/3} + C$   
**1824.**  $\frac{1}{525}(5x-4)^6(15x+2) + C$   
**1825.**  $\frac{1}{2} \sec 2u + C$   
**1826.**  $\frac{1}{2}u + \frac{1}{28} \sin 14u + C$   
**1827.**  $\frac{1}{3} \ln|\sin 3x| + C$   
**1828.**  $\frac{1}{18}(e^{3x} - 5)^6 + C$   
**1829.**  $\frac{1}{8}(3x^2 - 1)^{4/3} + C$   
**1830.**  $ex + C$   
**1831.**  $u^3 - 3u^{2/3} + C$   
**1832.**  $-\frac{1}{3}e^{\cos 3x} + C$   
**1833.**  $2 \ln|x+3| + C$   
**1834.**  $\frac{3^{2a}}{\ln 9} + C$   
**1835.**  $\sin 5x + C$   
**1836.**  $\frac{1}{4} \cos 4x + C$   
**1837.**  $x^2 + 3x - 2 \ln|x| + C$



- 1838.**  $\ln|5 + \tan x| + C$   
**1839.**  $\frac{3}{8}(3x^2 - 2)^{4/3} + C$   
**1840.**  $\frac{1}{96}(8z + 16)^{12} + C$   
**1841.**  $\frac{2}{3}(x + 2)^{3/2} + C$   
**1842.**  $-\frac{1}{6}\cos 6y + C$   
**1843.**  $\frac{1}{2}\sin 2x + C$   
**1844.**  $\frac{1}{2}\ln|\sec 2x + \tan 2x| + C$   
**1845.**  $\frac{1}{24}\sin^6 4x + C$   
**1846.**  $\ln|\sin a| + C$   
**1847.**  $\sin 2x + C$   
**1848.**  $\frac{4}{5}(x - 3)^{3/2}(x + 2) + C$   
**1850.**  $\frac{-1}{6(3y^2 + 2)^4} + C$   
**1851.**  $\frac{1}{10}e^{5x^2} + C$   
**1852.**  $\frac{1}{5}\sin 5y + C$   
**1853.**  $\frac{1}{2}x + \frac{1}{20}\sin 10x + C$   
**1854.**  $\frac{1}{15}\sin^3 5x + C$   
**1855.**  $\frac{1}{5}\ln|5x^2 - 3| + C$   
**1856.**  $\frac{1}{3}\ln|x| + C$   
**1857.**  $-\frac{1}{5}\cos(5\theta - 3\pi) + C$   
**1858.**  $\frac{4^y \ln 5}{\ln 4} + C$   
**1859.**  $\frac{1}{6}\ln|\sin(3x^2)| + C$   
**1860.**  $x + \frac{7}{2}e^{-2x} + C$   
**1861.**  $-\frac{1}{\ln 3}\cos(3^x) + C$   
**1862.**  $\sin x - \frac{1}{5}\sin^5 x + C$   
**1863.**  $\ln|x - 3| + C$   
**1864.**  $-\frac{5}{3}\ln|2 - 3x| + C$   
**1865.**  $-\frac{2}{3}\ln|2 - 3y^2| + C$   
**1866.**  $\frac{5}{6}\ln|1 + 2z^3| + C$   
**1867.**  $\frac{1}{2}\ln|3z^2 - 4z| + C$   
**1868.**  $-\frac{2}{3}\ln|1 + \cos 3\theta| + C$   
**1869.**  $-\frac{1}{2}\ln|3 - \sin 4\phi| + C$   
**1870.**  $\frac{1}{5}(\ln x)^5 + C$   
**1871.**  $\frac{1}{4}(\ln x)^4 + C$   
**1872.**  $\frac{1}{7}\sin^7 x + C$   
**1873.**  $-\frac{1}{9}\cos^3 3x + C$   
**1874.**  $\frac{1}{2}x - \frac{1}{16}\sin 8x + C$   
**1875.**  $-\frac{1}{5}\cos 5x + \frac{1}{15}\cos^3 5x + C$   
**1876.**  $-\frac{1}{24}\cos^4(2x)(2\sin^2(2x) + 1) + C$   
**1877.**  $\frac{1}{2}\tan 2x - x + C$   
**1878.**  $\frac{1}{2}\tan(e^{2x}) - \frac{1}{2}e^{2x} + C$   
**1879.**  $\frac{1}{2}\ln|x| + C$   
**1880.** 9  
**1881.**  $\frac{14}{\ln 2}$   
**1882.**  $2\sqrt{\sqrt{3} + 4} - 2\sqrt{\sqrt{2} + 4}$   
**1883.**  $\frac{671}{3240000}$   
**1884.**  $10e(e - 1)$   
**1885.**  $\frac{20}{3}$   
**1886.**  $\frac{2}{5}(e^5 - 1)$   
**1887.**  $4 - 4e^{-4/5}$   
**1888.**  $\frac{28}{9}(5\sqrt{5} - 2\sqrt{2})$   
**1889.**  $40(e^{1/20} - 1)$   
**1890.** 0  
**1891.** net and total are  $\frac{1}{3}(e^6 - 1)$   
**1892.** net and total are  $e^{12} - 1$   
**1893.** net is 0, total is 2  
**1894.** net is  $\frac{4}{3}$ , total is 2  
**1895.**  $\frac{16}{3}$   
**1896.**  $\frac{19}{\ln(\frac{3}{2})}$   
**1897.** 0  
**1898.**  $\frac{\ln 4}{\pi}$   
**1899.** (a) 7 (b) 13 (c) 10 (d) 10  
**1900.** (a)  $\frac{88}{25}$  (b)  $\frac{48}{25}$  (c) 3 (d)  $\frac{8}{3}$   
**1901.** (a) 102 (b) 57 (c)  $\frac{651}{8}$  (d) 78  
**1902.** (a)-(d) 56  
**1903.**  $\frac{500}{3}$   
**1904.**  $\frac{2}{3}$   
**1905.** 2  
**1906.**  $\frac{1}{2}$   
**1907.** 0  
**1908.**  $\frac{125}{6}$   
**1909.** 12  
**1910.**  $\frac{19}{3} + 8\ln \frac{2}{3}$   
**1911.**  $\frac{\pi}{4} - \ln \sqrt{2}$   
**1912.**  $\frac{80}{3}$   
**1913.** 32  
**1914.**  $\frac{1}{10}(143 - 36\sqrt{3})$   
**1915.**  $\frac{1}{6}$   
**1916.**  $\frac{1}{4}$   
**1917.**  $\frac{1}{12}$   
**1918.**  $2 - \sqrt{2}$   
**1919.**  $\frac{8}{3}$   
**1920.**  $\frac{5}{6}\sqrt{5}$   
**1921.**  $\frac{1}{6}(1000 - 61\sqrt{61})$   
**1922.**  $\frac{8}{3}$   
**1923.**  $2 - \frac{2}{3}\sqrt{3}$   
**1924.**  $-2\sqrt{10x - 3}$   
**1925.**  $\frac{\ln(3\pi x)}{x}$   
**1926.** -2, 5  
**1927.**  $\arcsin x + C$   
**1928.**  $\ln|\ln x| + C$   
**1929.**  $\frac{2}{3}(e^{12} - 1)$   
**1930.**  $\frac{1}{2}(e^2 - 21 + 7\ln 7)$   
**1931.**  $\frac{27}{4}$   
**1932.**  $\frac{(\ln 2 + 1)^2 - 1}{2e}$   
**1933.**  $-\frac{125}{4}$   
**1934.**  $2\sqrt{2}$   
**1935.** (a)  $\frac{5}{3}$  (b)  $\frac{17}{12}$  (c)  $\ln 4$  (d)  $2\pi$  (e)  $4\pi$  (f)  $4 + \int_1^2 \sqrt{\frac{4}{x^4} + 1} dx$  (g) 5.441  
**1936.**  $\frac{\pi}{4}(4 - \pi)$   
**1940.** (e) 5.105 (i) between 16 and 17 yrs (j) between 14.2 and 14.3 yrs  
**1941.**  $\frac{6}{5}x^{5/2} - 4\sqrt{x} + \frac{10}{3}x^{3/2} + C$   
**1942.**  $\frac{1}{2}z^2 + C$   
**1943.**  $\frac{2}{55}(5x^2 - 3)^{11} + C$

- 1944.**  $\frac{5^{3x} \ln 7}{3 \ln 5} + C$   
**1945.**  $\frac{5}{8}e^{4t^2} + C$   
**1946.**  $\frac{1}{60}(4x-1)^{3/2}(6x+1) + C$   
**1947.**  $3 \ln |z| + C$   
**1948.**  $-\frac{3}{2} \cos 2z + C$   
**1949.**  $\frac{4}{3} \sin 3z + C$   
**1950.**  $-2\sqrt{1-w^2} + C$   
**1951.**  $\frac{3}{2}x^4 - \frac{7}{2}x^{-2} + \frac{1}{3}x^3 + C$   
**1952.**  $\frac{1}{2} \ln |x^2 - 1| + C$   
**1953.**  $-e^{-x} + C$   
**1954.**  $-e^{1/x} + C$   
**1955.**  $\ln(e^x + 1) + C$   
**1956.**  $\frac{1}{3} \sin^3 x + C$   
**1957.**  $\frac{1}{2} \ln |\sin(x^2)| + C$   
**1958.**  $\frac{4}{9}(x^3 + 2)^{3/4} + C$   
**1959.**  $\frac{3}{4}(x^2 + 6x)^{2/3} + C$   
**1960.**  $x + \frac{1}{x+1} + C$   
**1961.**  $\frac{1}{3}(2y - 3y^2)^{3/2}$   
**1962.**  $\frac{2}{3} \ln |1 + 3u| + C$   
**1963.**  $\frac{1}{2} \arctan 2x + C$   
**1964.**  $\frac{-1}{8(1+4x^2)} + C$   
**1965.**  $2 \ln |x+1| + \frac{1}{2}x^2 + \frac{1}{x+1} - 2x + C$   
**1966.**  $\frac{1}{2}x + \frac{1}{4} \sin 2x + C$   
**1967.**  $\frac{1}{3} \tan 3u + C$   
**1968.**  $-\ln |\cos \theta| + C$   
**1969.**  $\frac{1}{2} \ln |1 - \cos 2t| + C$   
**1970.**  $\frac{1}{2} \ln |x(x-2)| + C$   
**1971.**  $-e^{-x}(x+1) + C$   
**1972.**  $\frac{1}{2}(\ln v)^2 + C$   
**1973.**  $\frac{1}{3}u^3 + C$   
**1974.**  $\frac{4}{3}$   
**1975.**  $\frac{10}{9}$   
**1976.** 2  
**1977.**  $\frac{2(e^{5/2} - 1)}{e^{3/2}}$   
**1978.** 0  
**1979.** 2  
**1980.** 6  
**1981.**  $-\frac{116}{15}$   
**1982.**  $\frac{1}{2}(e^3 - 1)$   
**1983.**  $1 - \frac{\pi}{4}$   
**1984.**  $2 - \sqrt{3}$   
**1985.** 1  
**1986.**  $\frac{3}{2}$   
**1987.** 2  
**1988.**  $\frac{1}{2}$   
**1989.**  $\frac{50}{3}$   
**1990.** 1  
**1991.** 1  
**1992.**  $\ln \sqrt{3}$   
**1993.**  $\frac{1}{9}(2e^3 + 1)$   
**1994.**  $2\sqrt{6} \arctan(\frac{\sqrt{6}}{3}) + \ln 100 - 4$   
**1995.**  $2e^2 - e + 2 - \ln 16$   
**1996.**  $\ln(\sqrt{10} - 1)$   
**1997.**  $\log e$   
**1998.** 1  
**1999.** divergent  
**2000.** divergent  
**2001.** 10  
**2002.** divergent  
**2003.** 1  
**2004.** divergent  
**2005.**  $\frac{1}{k \ln k}$   
**2006.** (a)-(b) positive (c) 0 (d)-(f) negative  
**2007.** (a) -27 (b) 24 (c) 0 (d) 450  
**2008.** 1.408  
**2009.** (a)  $y = -15x + 3$  (b)  $y = \frac{1}{30}(2-x)$  (c) decreasing (d) 0 and 2 (e) 0 (f) 54 (g) -84 (h) 24  
**2010.**  $y = \frac{\pi}{2} - x$   
**2011.** 40  
**2012.** (b)  $e^2 - \ln 4 + 1$  (c)  $\approx 73.564$  (d) 19.668 (e) 7.723  
**2013.** 2.899  
**2024.** divergent  
**2025.** divergent  
**2026.** divergent  
**2027.**  $\frac{2}{e-2}$   
**2028.** divergent  
**2029.**  $\frac{1}{12}$   
**2030.** divergent  
**2031.** divergent  
**2032.** convergent  
**2033.** divergent  
**2034.** convergent  
**2035.** convergent  
**2036.** convergent  
**2037.** divergent  
**2038.** convergent  
**2039.** convergent  
**2040.**  $\frac{1}{7}$   
**2041.**  $\frac{1}{e^x - 1}$   
**2042.** only one is false  
**2044.** (a)  $\frac{1}{3}$  (b) no (c) no  
**2045.** (a) 0 (b) no (c) yes  
**2046.** 15  
**2047.**  $\frac{172}{333}$   
**2048.**  $\frac{1}{4} < x < \frac{5}{4}$ ,  
sum is  $\frac{8}{8 - (4x - 3)^3}$   
**2049.**  $-\frac{\pi}{6} \leq x \leq \frac{\pi}{6}$   
**2050.**  $\frac{4}{3} \leq x < \frac{8}{3}$   
**2051.** 20 ft  
**2052.**  $\frac{19}{20}$   
**2053.** (a) divergent (b) convergent (c) divergent  
**2054.** (a)  $\frac{3}{5}$  (b) 3 (c)  $\infty$   
**2055.** C  
**2056.** C  
**2057.** B  
**2058.** E

## Answers to Last Year's Tests

### Limits Test

- |      |       |       |
|------|-------|-------|
| 1. C | 6. B  | 11. C |
| 2. E | 7. A  | 12. A |
| 3. B | 8. E  | 13. C |
| 4. B | 9. D  | 14. C |
| 5. E | 10. B | 15. C |

1. a. Since

$$\begin{aligned} f(x) &= \frac{|x|(x-3)}{9-x^2} = \frac{|x|(x-3)}{(3-x)(3+x)} \\ &= \frac{-|x|}{3+x}, \end{aligned}$$

we have that both 3 and  $-3$  are not in the domain; hence,  $D = \{x|x \neq \pm 3\}$ . The zeros are clearly 0 and 3, but 3 is not in the domain; hence, the only zero is 0.

- b.

$$\lim_{x \rightarrow 3} \frac{|x|(x-3)}{9-x^2} = \lim_{x \rightarrow 3} \frac{-|x|}{3+x} = -\frac{1}{2}.$$

- c. Clearly,  $x = -3$  is the only vertical asymptote since  $-3$  makes the denominator zero. To find the horizontal asymptotes, simply find the limits at positive and negative infinity:

$$\lim_{x \rightarrow \infty} \frac{|x|(x-3)}{9-x^2} = \lim_{x \rightarrow \infty} \frac{-|x|}{3+x} = -1$$

$$\lim_{x \rightarrow -\infty} \frac{|x|(x-3)}{9-x^2} = \lim_{x \rightarrow -\infty} \frac{-|x|}{3+x} = 1$$

So there are two horizontal asymptotes:  $y = 1$  and  $y = -1$ .

- d. Based on the previous parts, it should be easy to see that  $x = -3$  is an infinite discontinuity, and therefore is not removable. (Note that  $x = 3$  is a hole and so is removable.)
2. a. We have the following values:

$\frac{x}{x^x}$	1	0.5	0.4	0.3
$\frac{x}{x^x}$	1	0.707	0.693	0.697

$\frac{x}{x^x}$	0.2	0.1	0.01
$\frac{x}{x^x}$	0.725	0.794	0.955

- b. Judging from the data in the table, it appears as if both limits are 1. This is confirmed by the graphing calculator.

- c. Any answer between 0.697 and 0.707 is fine as long as you justify it using values in the table.

- d. The average rate of change is

$$\begin{aligned} \frac{g(0.4) - g(0.1)}{0.4 - 0.1} &= \frac{0.693 - 0.794}{0.3} \\ &= -0.337. \end{aligned}$$

3. a. This question becomes much simpler if you rewrite  $F$  as

$$\begin{aligned} (a^{-1} - x^{-1})^{-1} &= \left(\frac{1}{a} - \frac{1}{x}\right)^{-1} \\ &= \left(\frac{x-a}{ax}\right)^{-1} \\ &= \frac{ax}{x-a}. \end{aligned}$$

Then we can easily see that the domain  $D = \{x|x \neq 0, x \neq a\}$  and that there are no zeros.

- b. Since  $x = a$  is not in the domain,  $x = a$  is the vertical asymptote. Since the degree of the numerator is equal to the degree of denominator, we have  $y = a$  as the horizontal asymptote. The discontinuities are the infinite discontinuity at  $x = a$  and the removable discontinuity at  $x = 0$ .
- c.  $\lim_{x \rightarrow 0} F(x) = 0$ ;  $\lim_{x \rightarrow \infty} F(x) = a$ ; and  $\lim_{x \rightarrow a} F(x)$  does not exist.
- d. Solve  $\frac{6a}{6-a} = 12$  to get  $a = 4$ .

## Derivatives Test

- |      |       |       |
|------|-------|-------|
| 1. D | 6. C  | 11. D |
| 2. D | 7. D  | 12. D |
| 3. A | 8. B  | 13. D |
| 4. C | 9. C  | 14. D |
| 5. A | 10. D | 15. E |

1. a. Taking the derivative implicitly, we have

$$\begin{aligned}y' - y' \sin y &= 1 \\y'(1 - \sin y) &= 1 \\y' &= \frac{1}{1 - \sin y}\end{aligned}$$

- b. Vertical tangents have an undefined slope. Hence, we set the denominator of  $y'$  equal to zero and solve to get  $\sin y = 1$ , or  $y = \pi/2$ . Now we find the  $x$  value when  $y = \pi/2$ :

$$\begin{aligned}\frac{\pi}{2} + \cos \frac{\pi}{2} &= x + 1 \\ \frac{\pi}{2} &= x + 1 \\ x &= \frac{\pi}{2} - 1\end{aligned}$$

Hence, the vertical tangent is  $x = \frac{\pi}{2} - 1$ .

- c. We find the second derivative implicitly.

$$y'' = -\frac{y' \cos y}{(1 - \sin y)^2}$$

Now plug in the expression for  $y'$ .

$$\begin{aligned}y'' &= -\frac{\frac{1}{(1 - \sin y)} \cos y}{(1 - \sin y)^2} \\ &= -\frac{\cos y}{(1 - \sin y)^3}\end{aligned}$$

2. a. The volume is  $V = Bh$ , where  $B$  is the area of the triangular base. Hence,  $V = (\frac{1}{2}(3)(2))(5) = 15$ .
- b. By similar triangles, we have

$$\frac{\text{base of triangle}}{\text{height of triangle}} = \frac{2}{3},$$

or  $b = \frac{2}{3}h$ ; so that

$$V = \frac{1}{2} \left( \frac{2}{3}(h)(h) \right) (5) = \frac{5}{3}h^2.$$

When the trough is  $\frac{1}{4}$  full by volume, we have  $\frac{15}{4} = \frac{5}{3}h^2$ , so  $h = \frac{3}{2}$  at this instant. Now, we find the implicit derivative with respect to  $t$ :

$$\frac{dV}{dt} = \frac{10}{3}h \frac{dh}{dt}$$

and plug in our value of  $h$ :

$$\begin{aligned}-2 &= \frac{10}{3} \cdot \frac{3}{2} \cdot \frac{dh}{dt} \\ \frac{dh}{dt} &= -\frac{2}{5}\end{aligned}$$

- c. The area of the surface is  $A = 5b = 5 \cdot \frac{2}{3}h = \frac{10}{3}h$ . Finding the implicit derivative and using the value of  $dh/dt$  from part (b), we have

$$\begin{aligned}\frac{dA}{dt} &= \frac{10}{3} \cdot \frac{dh}{dt} \\ &= \frac{10}{3} \cdot \frac{-2}{5} = -\frac{4}{3}\end{aligned}$$

3. a. The domain is whatever makes  $x^4 - 16x^2 \geq 0$ , or  $x^2(x^2 - 16) \geq 0$ ; thus, we find have either  $x = 0$  or  $x^2 \geq 16$ . The domain is therefore  $(-\infty, -4) \cup \{0\} \cup (4, \infty)$ .

- b. We have

$$\begin{aligned}f(-x) &= \sqrt{(-x)^4 - 16(-x)^2} \\ &= \sqrt{x^4 - 16x^2} = f(x)\end{aligned}$$

so  $f$  is even.

- c. Observe:

$$\begin{aligned}f'(x) &= \frac{1}{2}(x^4 - 16x^2)^{-1/2}(4x^3 - 32x) \\ &= \frac{2x^3 - 16x}{\sqrt{x^4 - 16x^2}} = \frac{2x(x^2 - 8)}{|x|\sqrt{x^2 - 16}}\end{aligned}$$

- d. From part (c), we have

$$f'(5) = \frac{10(25 - 8)}{5\sqrt{25 - 16}} = \frac{34}{3}$$

so the slope of the normal is  $-\frac{3}{34}$ .

## Applications of Derivatives Test

1. D                    6. A                    11. B  
 2. D                    7. B                    12. D  
 3. D                    8. E                    13. D  
 4. C                    9. B                    14. C  
 5. D                    10. A                   15. A

1. a. We have

$$\begin{aligned} v(t) &= x'(t) = 2\pi - 2\pi \sin 2\pi t \\ &= 2\pi(1 - \sin 2\pi t) \end{aligned}$$

- b. We have

$$a(t) = v'(t) = x''(t) = -4\pi^2 \cos 2\pi t$$

- c. The particle is at rest when  $v(t) = 0$ :

$$\begin{aligned} 2\pi(1 - \sin 2\pi t) &= 0 \\ \sin 2\pi t &= 1 \\ 2\pi t &= \frac{\pi}{2} \\ t &= \frac{1}{4}, \frac{5}{4}, \frac{9}{4} \end{aligned}$$

- d. We find the critical points of  $v(t)$  by setting  $a(t) = 0$ :

$$\begin{aligned} -4\pi^2 \cos 2\pi t &= 0 \\ \cos \pi t &= 0 \\ \pi t &= \frac{\pi}{2}, \frac{3\pi}{2}, \dots \\ t &= \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \dots \end{aligned}$$

However,  $v(\frac{1}{4}) = 0$  and  $v(\frac{3}{4}) = 4\pi$  are the only possible maximum values (since all other odd multiples of  $\frac{1}{4}$  give values equivalent to these two). Thus, the maximum velocity is  $4\pi$ .

2. a. The absolute maximum occurs at  $x = -1$  because  $f$  is increasing on the interval  $[-3, -1]$  and decreasing on the interval  $[-1, 3]$ . The absolute minimum must occur at  $x = 1$  or at an endpoint. However,  $f$  is decreasing on the interval  $[-1, 3]$ ; therefore, the absolute minimum is at an endpoint. Since  $f(-3) = 4 > 1 = f(3)$ , the absolute minimum is at  $x = 3$ .

- b. There is an inflection point at  $x = 1$  because the graph changes from concave up to concave down (or  $f''$  changes from positive to negative) there.

3. a. We first find critical points:

$$\begin{aligned} f'(x) &= 3x^2 - 10x + 3 = 0 \\ (3x - 1)(x - 3) &= 0 \\ x &= \frac{1}{3} \text{ and } 3 \end{aligned}$$

Since  $f'$  is positive for  $x < \frac{1}{3}$  and for  $x > 3$ , the increasing intervals are  $(-\infty, \frac{1}{3})$  and  $(3, \infty)$ .

- b. Since  $f''(x) = 6x - 10$ , the inflection point is  $x = \frac{5}{3}$ . Thus, since  $f''$  is negative for  $x < \frac{5}{3}$ , the graph of  $f$  is concave down on  $(-\infty, \frac{5}{3})$ .

- c. From part (a), we know that  $x = 3$  gives the minimum value. Hence, we must have  $f(3) = 11$ :

$$\begin{aligned} f(3) &= 3^3 - 5(3^2) + 3(3) + k = 11 \\ -9 + k &= 11 \\ k &= 20 \end{aligned}$$

## Integrals Test

1. E                    6. D                    11. C  
 2. B                    7. A                    12. B  
 3. E                    8. B                    13. E  
 4. B                    9. C                    14. D  
 5. C                    10. B                    15. A

1. a. We have  $T(0) = -15$  and  $T(12) = 5$ .  
 This gives the system of equations

$$\begin{aligned} -A - B &= -15 \\ -A + B &= 5 \end{aligned}$$

Hence,  $A = 5$  and  $B = 10$ .

b.

$$\begin{aligned} \frac{1}{10} \int_0^{10} \left( -5 - 10 \cos\left(\frac{\pi h}{12}\right) \right) dh \\ = -6.910 \end{aligned}$$

c.

$$\begin{aligned} \int_6^{10} T(h) dh \\ = \frac{1}{2} [T(6) + 2T(7) + 2T(8) \\ + 2T(9) + T(10)] \\ = \frac{1}{2} [-5 + 2(-2.412) + 2(0) \\ + 2(2.071) + 3.66] \\ = -1.011 \end{aligned}$$

This integral represents the average temperature in degrees Fahrenheit from 6 AM to 10 AM.

- d. Since  $T(h) = -5 - 10 \cos\left(\frac{\pi h}{12}\right)$ , we have

$$T'(h) = -\frac{5\pi}{6} \sin\left(\frac{\pi h}{12}\right).$$

2. Differentiating the expression in 1) gives  $f''(x) = 2ax + b$ . From 2) we have that  $f'(1) = 2a + b = 6$  and  $f''(1) = 2a + b = 18$ . Thus we have a system of equations in  $a$

and  $b$  that we can easily solve to get  $a = 12$  and  $b = -6$ . Therefore

$$f'(x) = 12x^2 - 6x,$$

and

$$f(x) = \int (12x^2 - 6x) dx = 4x^3 - 3x^2 + C.$$

Using 3) we can solve for  $C$ :

$$\begin{aligned} 18 &= \int_1^2 f(x) dx = x^4 - x^3 + Cx \Big|_1^2 \\ &= 16 - 8 + 2C - (1 - 1 + C) = 8 + C \end{aligned}$$

thus,  $C = 10$ , and  $f(x) = 4x^3 - 3x^2 + 10$ .

3. a.

$$a(t) = v'(t) = -2\pi \cos(2\pi t)$$

- b. Set  $v(t) = 0$  and solve.

$$\begin{aligned} 1 - \sin(2\pi t) &= 0 \\ \sin(2\pi t) &= 1 \\ 2\pi t &= \frac{\pi}{2} \\ t &= \frac{1}{4}, \frac{5}{4} \end{aligned}$$

c.

$$\begin{aligned} x(t) &= \int v(t) dt \\ &= \int (1 - \sin(2\pi t)) dt \\ &= t + \frac{1}{2\pi} \cos(2\pi t) + C \end{aligned}$$

Since  $x(0) = 0$ , we have

$$\begin{aligned} 0 &= 0 + \frac{\cos 0}{2\pi} + C \\ 0 &= \frac{1}{2\pi} + C \\ C &= -\frac{1}{2\pi} \end{aligned}$$

Thus,  $x(t) = t + \frac{1}{2\pi} \cos(2\pi t) - \frac{1}{2\pi}$ .

## Applications of Integrals Test

1. A                      6. D                      11. B  
 2. B                      7. A                      12. B  
 3. E                      8. A                      13. D  
 4. A                      9. C                      14. B  
 5. E                      10. D                     15. C

1. a. First, we find the  $x$ -coordinates of the intersection points of the two graphs. Set them equal and solve, using your calculator:

$$4e^{-x} = \tan\left(\frac{x}{2}\right)$$

$$x = 1.4786108$$

Let  $a = 1.4786108$ . Thus, the area  $A$  is

$$A = \int_0^a \left(4e^{-x} - \tan\left(\frac{x}{2}\right)\right) dx = 2.483$$

- b. The volume  $V$  is

$$V = \pi \int_0^a \left[ \left(4e^{-x}\right)^2 - \left(\tan\left(\frac{x}{2}\right)\right)^2 \right] dx$$

$$= 7.239\pi = 22.743$$

- c. Since the diameter is in  $R$ , the length of the radius is  $\frac{1}{2} \left[4e^{-x} - \tan\left(\frac{x}{2}\right)\right]$ . The area of a semicircle with radius  $r$  is  $A = \frac{1}{2}\pi r^2$ . Hence,

$$A = \frac{\pi}{2} \left( \frac{1}{2} \left[4e^{-x} - \tan\left(\frac{x}{2}\right)\right] \right)^2$$

$$= \frac{\pi}{8} \left[4e^{-x} - \tan\left(\frac{x}{2}\right)\right]^2.$$

Therefore, the volume  $V$  is

$$V = \int_0^a \frac{\pi}{8} \left[4e^{-x} - \tan\left(\frac{x}{2}\right)\right]^2 dx$$

$$= 0.755\pi = 2.373$$

2. a. You should have segments of zero slope at the three points where  $x = 0$ . You should have negative slopes with increasing steepness bottom to top at the points where  $x = -1$ . Finally, you should have positive slopes with increasing steepness from bottom to top and from left to right at the points where  $x = 1$  and  $x = 2$ .

- b. You should draw a graph that is concave up, decreasing for  $x < 0$ , increasing for  $x > 0$ , and that passes through the point  $(0, 2)$ .

- c. To solve, we separate and integrate:

$$\frac{dy}{dx} = \frac{xy}{2}$$

$$\int \frac{dy}{y} = \int \frac{x}{2} dx$$

$$\ln y = \frac{1}{4}x^2 + C$$

$$y = Ce^{x^2/4}$$

With the initial condition, we find that  $C = 2$ , so the equation is  $y = 2e^{x^2/4}$ . Therefore,  $y(2) = 2e^{4/4} = 2e = 5.4365$ .

3. a. Since  $v(1.5) = 1.167 > 0$  the particle is moving up the  $y$ -axis.

- b. The acceleration is

$$a(t) = v'(t) = \sin(t^2) + 2t^2 \cos(t^2)$$

so that  $a(1.5) = -2.049 < 0$ , which indicates the velocity is decreasing.

- c. We have

$$y(t) = \int v(t) dt = -\frac{\cos(t^2)}{2} + C$$

and using the initial condition  $y(0) = 3$ , we find  $C = \frac{7}{2}$ . Hence,

$$y(t) = \frac{7 - \cos(t^2)}{2}.$$

Therefore,  $y(2) = \frac{7 - \cos 4}{2} = 3.827$ .

- d. The total distance is given by

$$\int_0^2 |v(t)| dt = 1.173,$$

or

$$\int_0^{\sqrt{\pi}} v(t) dt - \int_{\sqrt{\pi}}^2 v(t) dt = 1.173.$$

## Techniques of Integration Test

1. C                    6. D                    11. B  
 2. B                    7. C                    12. C  
 3. C                    8. C                    13. C  
 4. C                    9. C                    14. C  
 5. E                    10. D                    15. C

1. a. The average value of  $f$  from 0 to 3 is

$$\frac{1}{3} \int_0^3 f(x) dx = \frac{5-1}{2},$$

and solving for the integral gives

$$\int_0^3 f(x) dx = 6.$$

- b. Again, the average value of  $f$  from 0 to  $x$  is

$$\frac{1}{x} \int_0^x f(t) dt = \frac{5+f(x)}{2},$$

or

$$\int_0^x f(t) dt = \frac{5x + xf(x)}{2}.$$

Using the Fundamental Theorem to differentiate both sides, we have

$$f(x) = \frac{5}{2} + \frac{1}{2}f(x) + \frac{1}{2}xf'(x)$$

$$2f(x) = 5 + f(x) + xf'(x)$$

$$f'(x) = \frac{f(x) - 5}{x}.$$

- c. From part (b), we have a differential equation that can be solved.

$$\frac{dy}{dx} = \frac{y-5}{x}$$

$$\int \frac{dy}{y-5} = \int \frac{dx}{x}$$

$$\ln(y-5) = \ln x + C$$

$$y-5 = Cx$$

$$y = Cx + 5$$

and since  $f(3) = -1$ , we get that  $C = -2$ ; hence,  $y = f(x) = 5 - 2x$ .

2. a.

$$\begin{aligned} R &= \int_1^3 \ln x dx = (x \ln x - x) \Big|_1^3 \\ &= 3 \ln 3 - 2 = 1.296. \end{aligned}$$

- b.

$$V = \pi \int_1^3 (\ln x)^2 dx = 1.029\pi = 3.233$$

- c. We solve  $y = \ln x$  for  $x$  to get  $x = e^y$ . When  $x = 1$ ,  $y = 0$ , and when  $x = 3$ ,  $y = \ln 3$ . Thus,

$$V = \pi \int_0^{\ln 3} (3 - e^y) dy$$

3. a.

$$\frac{dy}{dx} = \frac{-xy}{\ln y}$$

$$\int \frac{\ln y}{y} dy = \int -x dx$$

$$\frac{(\ln y)^2}{2} = \frac{-x^2}{2} + C$$

$$(\ln y)^2 = -x^2 + C$$

$$\ln y = \pm \sqrt{C - x^2}$$

$$y = e^{\pm \sqrt{C - x^2}}$$

- b. We find  $C$ .

$$y = e^{\pm \sqrt{C - x^2}}$$

$$e^2 = e^{\pm \sqrt{C}}$$

$$2 = \pm \sqrt{C}$$

$$C = 4$$

so that  $y = e^{\pm \sqrt{4 - x^2}}$ .

- c. If  $x = 2$ , then  $y = 1$  and  $\ln y = 0$ . This causes the derivative  $\frac{-xy}{\ln y}$  to be undefined.



## Series, Vectors, Parametric, and Polar Test

- |      |       |       |
|------|-------|-------|
| 1. E | 6. C  | 11. D |
| 2. D | 7. D  | 12. C |
| 3. B | 8. D  | 13. E |
| 4. A | 9. E  | 14. D |
| 5. A | 10. C | 15. A |

1. a.

$$\mathbf{v}(t) = \left\langle -\frac{3\pi}{4} \sin \frac{\pi t}{4}, \frac{5\pi}{4} \cos \frac{\pi t}{4} \right\rangle$$

$$\mathbf{v}(3) = \left\langle -\frac{3\pi\sqrt{2}}{8}, -\frac{5\pi\sqrt{2}}{8} \right\rangle$$

$$\begin{aligned} \|\mathbf{v}(3)\| &= \sqrt{\frac{18\pi^2}{64} + \frac{50\pi^2}{64}} \\ &= \frac{\pi\sqrt{17}}{4} = 1.031\pi = 3.238 \end{aligned}$$

b.

$$\mathbf{a}(t) = \left\langle -\frac{3\pi^2}{16} \cos \frac{\pi t}{4}, -\frac{5\pi^2}{16} \sin \frac{\pi t}{4} \right\rangle$$

$$\begin{aligned} \mathbf{a}(3) &= \left\langle \frac{3\pi^2\sqrt{2}}{32}, -\frac{5\pi^2\sqrt{2}}{32} \right\rangle \\ &= \langle 0.133\pi^2, -0.221\pi^2 \rangle \\ &= \langle 1.309, -2.181 \rangle \end{aligned}$$

c. Since

$$\sin^2 \theta + \cos^2 \theta = 1,$$

we have, upon solving  $x(t)$  and  $y(t)$ 

for the trigonometric terms,

$$\frac{x^2}{3} + \frac{y^2}{5} = 1.$$

2. a. This curve is a 3-petal rose with petal tips at Cartesian coordinates
- $(\sqrt{3}, 1)$
- ,
- $(-\sqrt{3}, 1)$
- , and
- $(0, -2)$
- .

b.

$$\frac{1}{2} \int_0^\pi (2 \sin 3\theta)^2 d\theta = \pi = 3.142$$

c.

$$\begin{aligned} \frac{dy}{dx} &= \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta} \\ &= \frac{6 \cos 3\theta \sin \theta + 2 \sin 3\theta \cos \theta}{6 \cos 3\theta \cos \theta - 2 \sin 3\theta \sin \theta} \\ \left. \frac{dy}{dx} \right|_{\theta=\pi/4} &= \frac{1}{2} \end{aligned}$$

3. a.

$$f(x) \approx 5 - 3x + \frac{x^2}{2} + \frac{4x^3}{6}$$

b.

$$g(x) \approx 5 - 3x^2 + \frac{x^4}{2}$$

c.

$$h(x) \approx 5x - \frac{3x^2}{2} + \frac{x^3}{6}$$

- d.
- $h(1) = \int_0^1 f(t) dt$
- , but the exact value cannot be determined since
- $f(t)$
- is only known at
- $t = 0$
- and
- $t = 1$
- .

The necessary is possible.  
The optional is expensive.  
The arbitrary is unlikely.  
Discard the superfluous.

—*Robert Fripp*